Magnetism in matter

Magnetism in matter

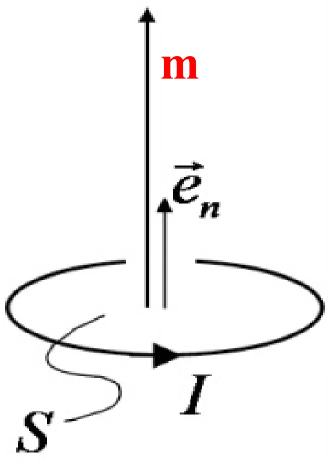
- The magnetic dipole
- Electron and nuclear magnetic moment
- Diamagnetism, paramagnetism, ferromagnetism
- Magnetostatic calculations
- Forces, torques, magnetic levitation
- Galvanomagnetic effects in matter

The magnetic dipole

$$\boxed{\mathbf{m} \triangleq \frac{1}{2}\int_V \mathbf{r}\times\mathbf{J}(\mathbf{r})dV}$$

m: Magnetic dipole moment (magnetic dipole)

Example 1. Circular electrical circuit with current *I*:

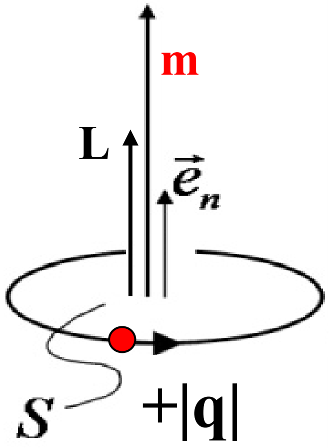


$$\mathbf{m} = \frac{1}{2}\int_V \mathbf{r}\times\mathbf{J}(\mathbf{r})dV = \frac{1}{2}I\oint_C \mathbf{r}\times d\mathbf{l} = \frac{1}{2}I\hat{\mathbf{e}}_n\oint_C rdl = \frac{1}{2}I\hat{\mathbf{e}}_n r\oint_C dl$$

$$= \frac{1}{2}Ir2\pi r\hat{\mathbf{e}}_n = IS\hat{\mathbf{e}}_n \quad [\mathrm{A\ m^2}]$$

Example 2. Charged particle with mass m_p in circular orbit:

Magnetic moment as a function of equiv. current and as a function of angular momentum:



The equivalent current is

$$I = q\frac{1}{T} = q\frac{v}{2\pi R} = q\frac{\omega}{2\pi}$$

$$\Rightarrow$$

$$\mathbf{m} = IS\hat{\mathbf{e}}_n = q\frac{v}{2}R\hat{\mathbf{e}}_n = q\frac{\omega}{2\pi}S\hat{\mathbf{e}}_n$$

The angular moment is:

$$\mathbf{L} = \mathbf{r}\times(m_p\mathbf{v}) = Rm_p v\hat{\mathbf{e}}_n = m_p\omega R^2\hat{\mathbf{e}}_n$$

$$\Rightarrow$$

$$\mathbf{m} = q\frac{1}{2\pi}\frac{\mathbf{L}}{m_p R^2}S = \frac{q}{2m_p}\mathbf{L}$$

m: Magnetic dipole moment (magnetic dipole)

Note: We can call this magnetic dipole moment of the particle the "orbital" magnetic dipole moment of the particle, to distinguish it from the intrinsic magnetic dipole moment or spin that some particles (electrons, protons, ..) possess.

Magnetic dipoles of atoms and molecules

The magnetic dipole moments
of atoms and molecules arise from:

1. the intrinsic (spin) angular momentum of the electrons, protons, and neutrons
2. the orbital angular momentum of the electrons

Intrinsic magnetic moment (or spin)

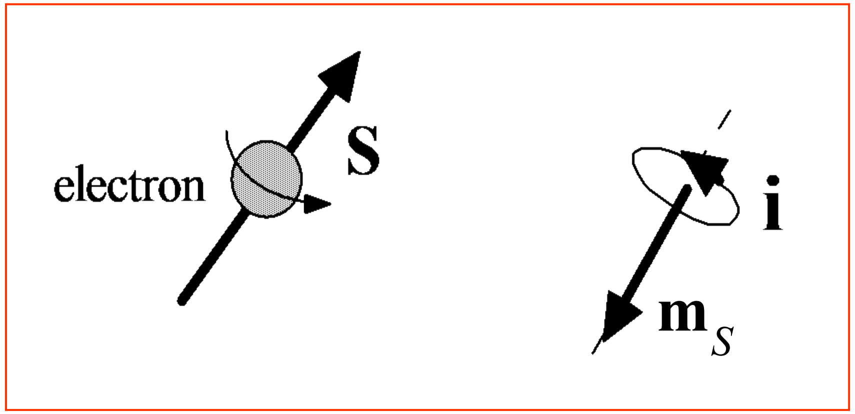
No "classical physics" analogy:

$$\mathbf{m}_S = g\frac{q}{2m}\mathbf{S} \qquad (\mathbf{S}: \text{spin angular momentum})$$

For the electron: S=1/2

$$m_S = 2\mu_B\sqrt{S(S+1)} = \mu_B\sqrt{3} \approx \mu_B$$

(the classical picture of the electron rotates about itself is not correct)



Orbital magnetic moment

$$\mathbf{m} \equiv \frac{1}{2}\int_V \mathbf{r}\times\mathbf{J}(\mathbf{r})dV = \frac{1}{2}\sum_{k=1}^{N} q_k(\mathbf{r}_k\times\mathbf{v}_k) \qquad \mathbf{L}_k \equiv m_k\left(\mathbf{r}_k\times\mathbf{v}_k\right) \qquad (\mathbf{L}_k : \text{orbital angular momentum of the particle } k)$$

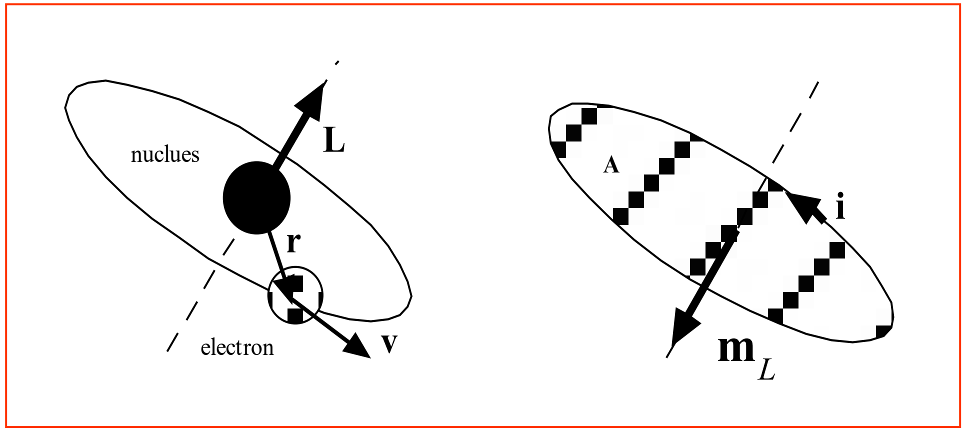
$$\Rightarrow \quad \mathbf{m}_L = \frac{1}{2}\sum_{k=1}^{N}\frac{q_k}{2m_k}\mathbf{L}_k$$

If all particules have the same q/m ratio:

$$\mathbf{m}_L = \frac{q}{2m}\sum_{k=1}^{N}\mathbf{L}_k = \frac{q}{2m}\mathbf{L} \qquad (\mathbf{L} = \sum_{k=1}^{N}\mathbf{L}_k \text{ is the total orbital angular momentum})$$

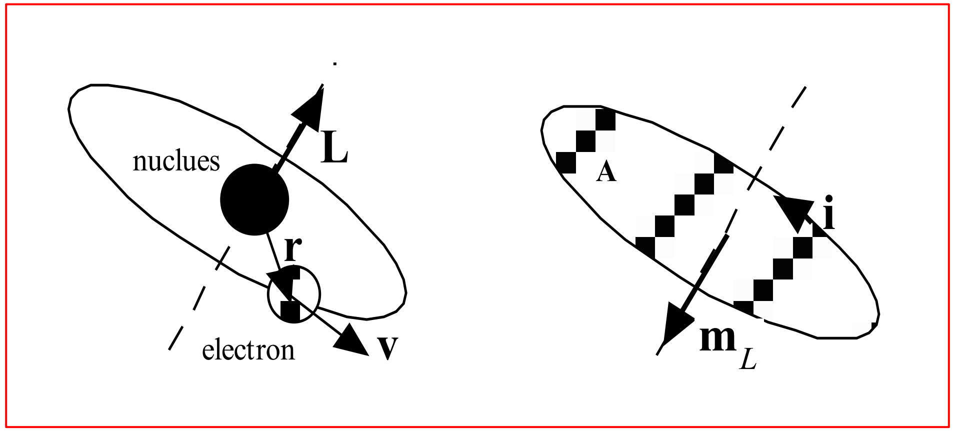
For the electron:

$$m_L = -\frac{e}{2m_e}L \approx \mu_B$$



Orbital magnetic moment of an electron:

classical computation



- Equilibrium between **Coulomb force** and **centrifugal force:**

$$\frac{e}{4\pi\varepsilon_0 r^2} = m_e \frac{v^2}{r} \quad \Rightarrow \quad v = e\sqrt{\frac{1}{4\pi\varepsilon_0 r}\frac{1}{m_e}}$$

- **Orbital angular momentum:**

$$L = r m_e v \qquad \Rightarrow (\text{for} \quad r \sim 10^{-10}\text{m}) \qquad L = e\sqrt{\frac{m_e r}{4\pi\varepsilon_0}} \cong 1\times 10^{-34}\ \text{Js} \cong \hbar$$

- The **orbital magnetic moment** is then**:**

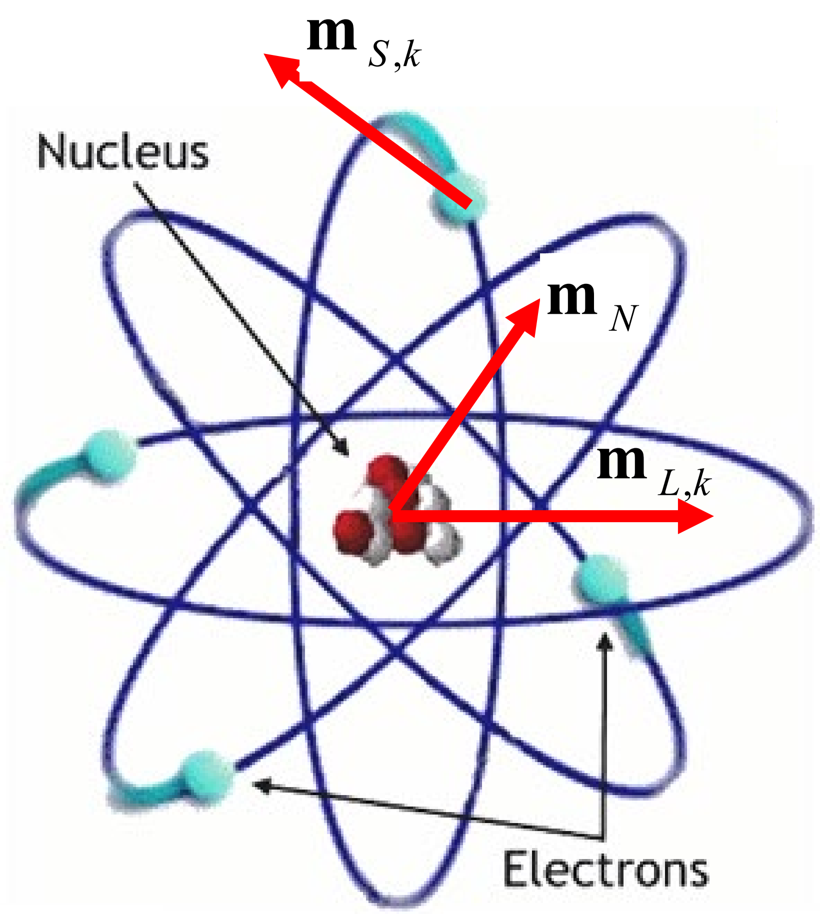
$$\mu_L = iA = -\left(\frac{ev}{2\pi r}\right)(\pi r^2) = -\frac{ev}{2} r = -\frac{e}{2m_e} L \approx \mu_B$$

$$i \sim \frac{\mu_B}{A} \sim \frac{\mu_B}{\pi r^2} \sim 0.3\ \text{mA}$$

where:

$$\mu_B = \frac{e\hbar}{2m_e} \cong 9.3\times 10^{-24} \quad \text{Am}^2$$ is called *Bohr magneton.*

Magnetic dipole of the atom



$\mathbf{m}_{L,k}$: Orbital magnetic moment of the electron k

$\mathbf{m}_{S,k}$: Intrinsic magnetic moment of the electron k

$\mathbf{m}_N$: Magnetic moment of the nucleus

$$\mathbf{m}_{S,k} \cong \mathbf{m}_{L,k} \qquad \mathbf{m}_N << \mathbf{m}_{S,k}, \mathbf{m}_{L,k}$$

Total magnetic moment of the atom i:

$$\boxed{|\mathbf{m}_i| \cong \alpha \mu_B \qquad \alpha \in [0,10]}$$

$$\mu_B = \frac{e\hbar}{2m_e} \cong 9.24 \times 10^{-24}\ \mathrm{Am}^2$$

Note:

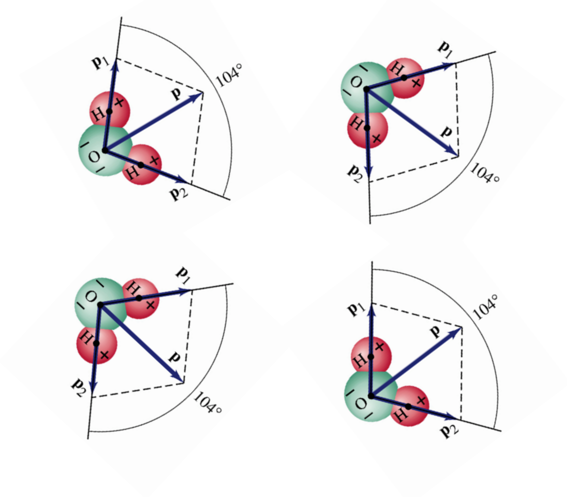
"equivalent atomic current"

$$I \approx \frac{\mu_B}{S} \approx \frac{10^{-23}\,\mathrm{Am}^2}{10^{-20}\,\mathrm{m}^2} = 10^{-3}\,\mathrm{A}$$

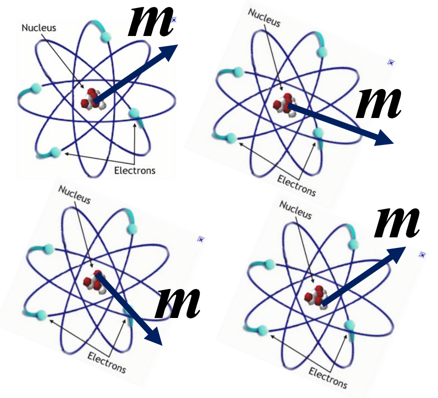
"equivalent atomic velocity"

$$v \approx \frac{2\pi R I}{e} \approx \frac{10^{-9}\,\mathrm{m} \times 10^{-3}\,\mathrm{A}}{10^{-19}\,\mathrm{C}} \approx 10^{7}\,\mathrm{m/s}$$

«Dielectric» materials:
composed by <u>microscopic</u> **electric dipoles**
(permanent or induced by an external electric field **E**)



Matériaux «magnétiques»:
composed by <u>microscopic</u> **magnetic dipoles**
(permanent or induced by an external magnetic field **B**)



<u>Microscopic:</u>
dimensions of an atom or a molecule

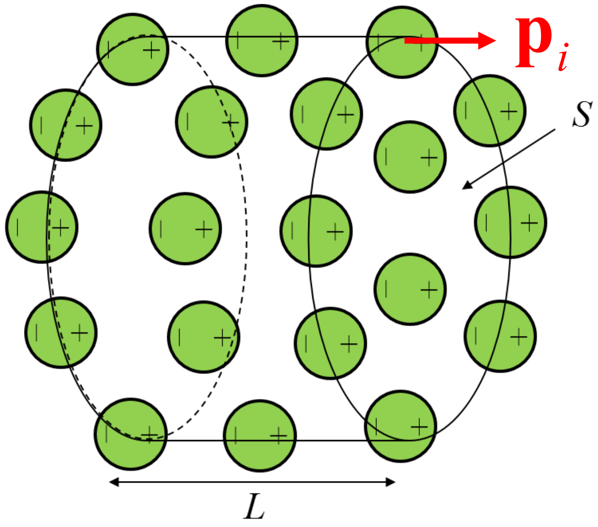
Polarization P and magnetization M

Polarization P

Electric dipole per unit of volume
(effective density of electric dipoles)

$$\mathbf{P} = \frac{\sum_i \mathbf{p}_i}{\Delta V} = \frac{\sum_i \mathbf{p}_i}{LS}$$

$$[\mathbf{p}_i] = \mathrm{Cm}; \quad [\mathbf{P}] = \mathrm{C/\,m^2}$$

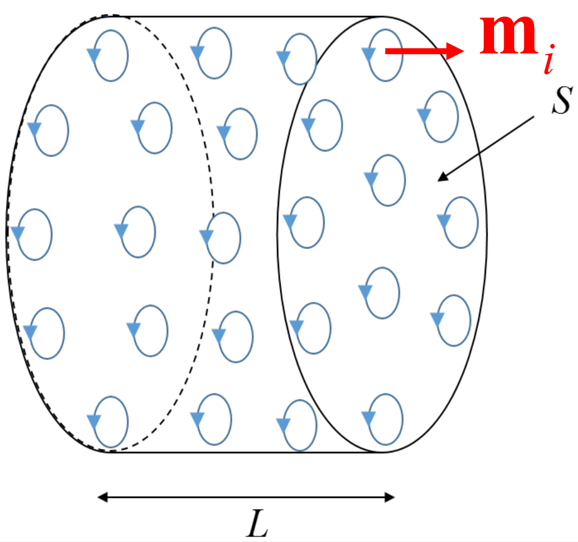


Magnetization M

Magnetic dipole per unit of volume
(effective density of magnetic dipoles)

$$\mathbf{M} = \frac{\sum_i \mathbf{m}_i}{\Delta V} = \frac{\sum_i \mathbf{m}_i}{LS}$$

$$[\mathbf{m}_i] = \mathrm{Am^2}; \quad [\mathbf{M}] = \mathrm{Am^{-1}}$$

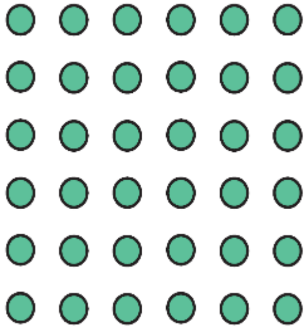
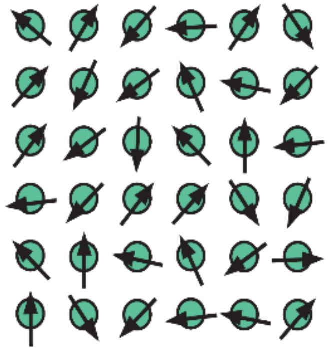
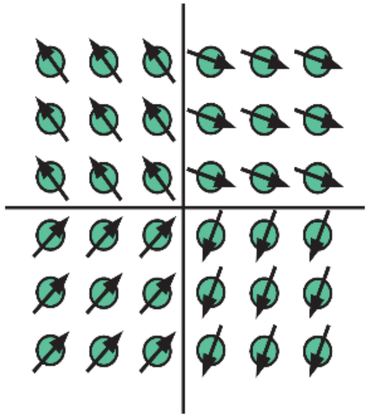
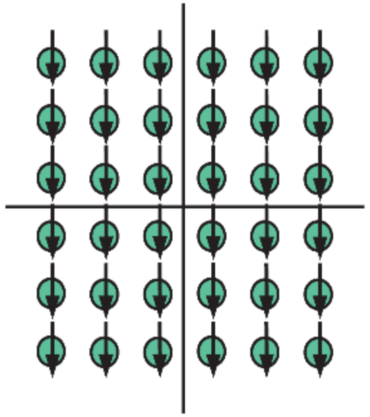
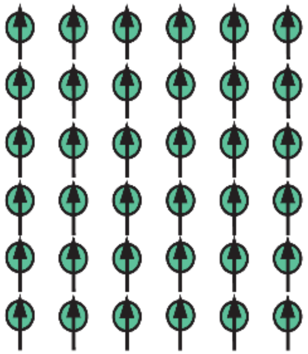
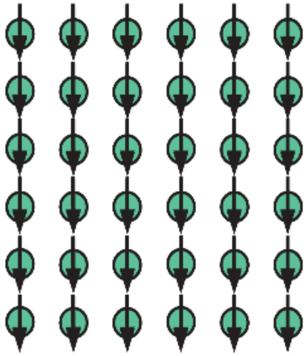
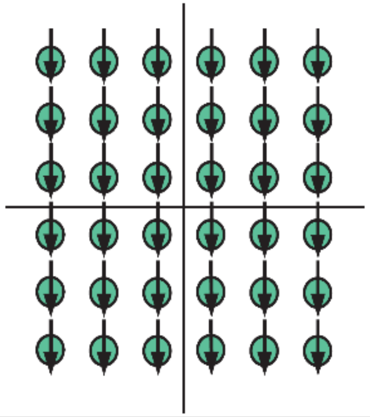
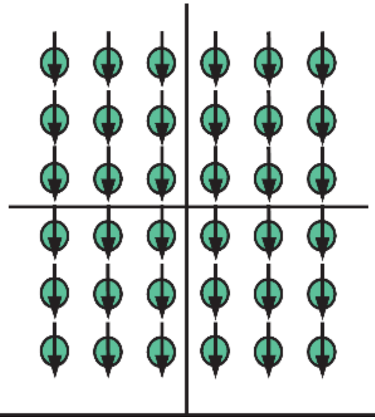


Note: In reality, the electric and magnetic dipoles are not all aligned in the same direction, except at very low temperatures, under very strong electric and magnetic fields, or in ferroelectric and ferromagnetic materials (as will be seen later in the course).

Types of magnetism

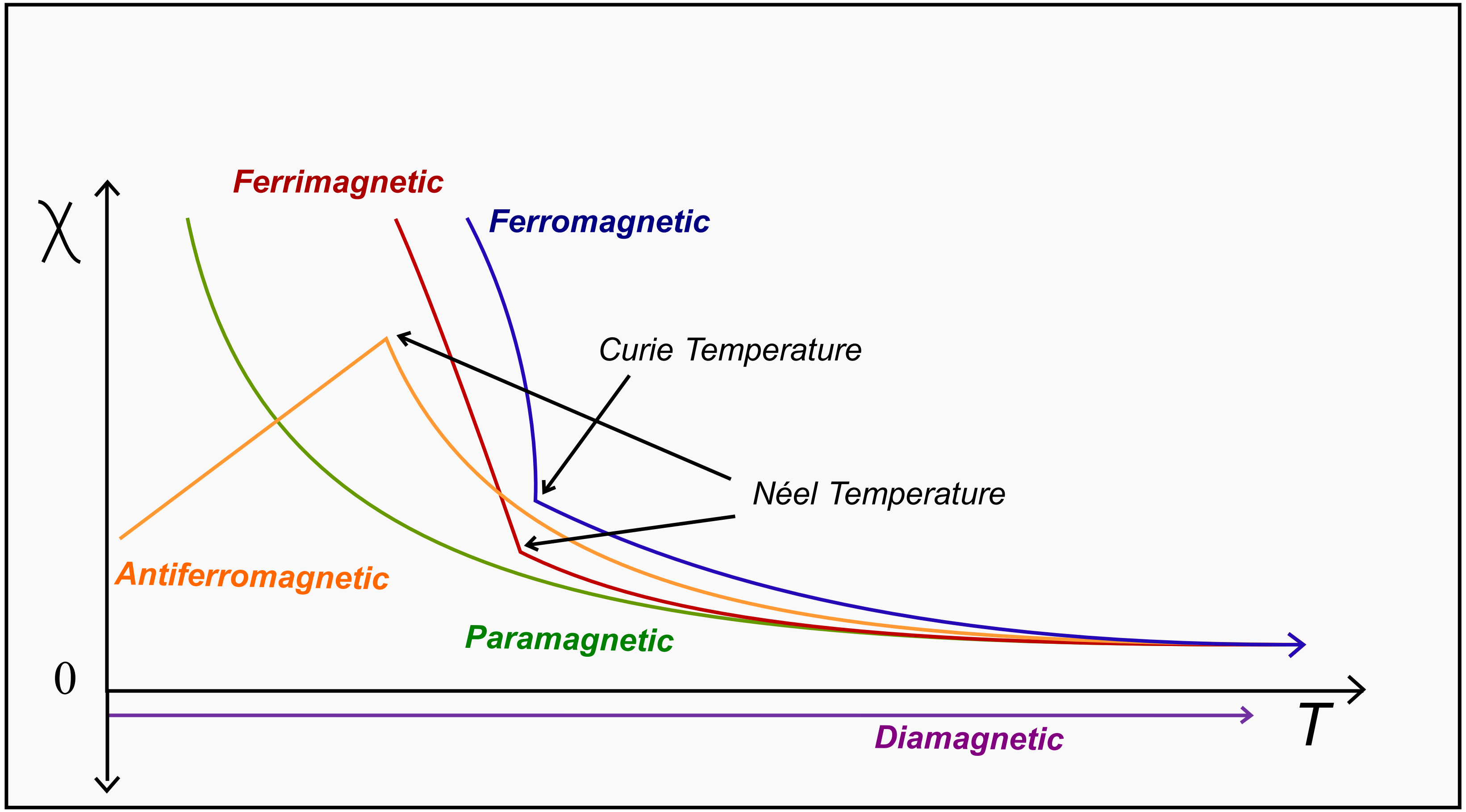
$$\mathbf{M} = \chi_m \mathbf{H} \qquad \chi_m : \text{ Magnetic susceptibility}$$

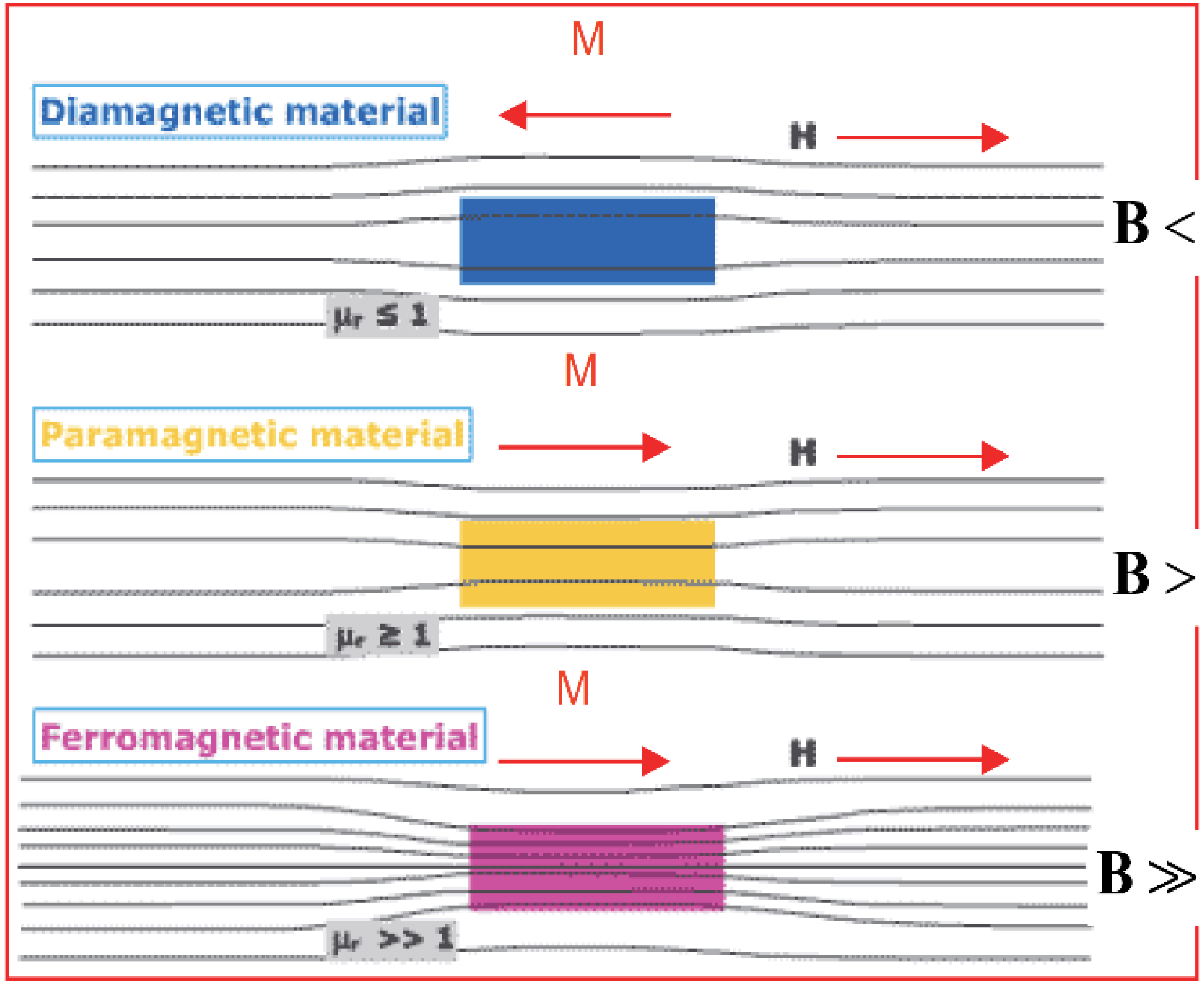
Type	Typical values	Physical Origin
Diamagnetism	$\chi \approx -10^{-6}$	Non-interacting «induced» magnetic dipoles
Paramagnetism	$\chi \approx +(10^{-5} \div 10^{-3})$	Orientation of non-interacting «permanent» magnetic dipoles
Superparamagnetism	$\chi \approx +10^{-2}$	Orientation of non-interacting «permanent» large (many atoms) magnetic dipoles
Ferromagnetism	$\chi = +\left(0 \div 10^{6}\right)$	Orientation + coupling + anisotropy of «permanent» magnetic dipoles

	DIAMAGNETIC	PARAMAGNETIC	FERROMAGNETIC SOFT (IDEAL)	FERROMAGNETIC HARD (IDEAL)
$\boxed{B = 0}$				
$\boxed{\mathbf{B} \quad \downarrow}$ $\boxed{mB >> kT}$				

Magnetic susceptibility vs temperature

Ferromagnetic ↑ ↑ ↑ ↑ ↑ ↑ ↑	Below T_C, spins are aligned parallel in magnetic domains
Antiferromagnetic ↑ ↓ ↑ ↓ ↑ ↓ ↑	Below T_N, spins are aligned antiparallel in magnetic domains
Ferrimagnetic ↑ ↓ ↑ ↓ ↑ ↓ ↑	Below T_C, spins are aligned antiparallel but do not cancel
Paramagnetic ↑ ↓ ↑ ↑ ↓ ↑ ↓	Spins are randomly oriented (any of the others above T_C or T_N)





$\mathbf{B} < \mu_0 \mathbf{H}$ (inside the material)

$$\boxed{\begin{array}{c}\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M} \\ \text{(everywhere)}\end{array}}$$

$\mathbf{B} > \mu_0 \mathbf{H}$ (inside the material)

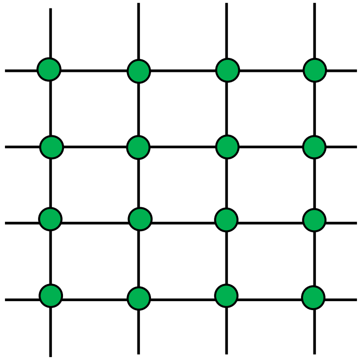
$\mathbf{B} \gg \mu_0 \mathbf{H}$ (inside the material)

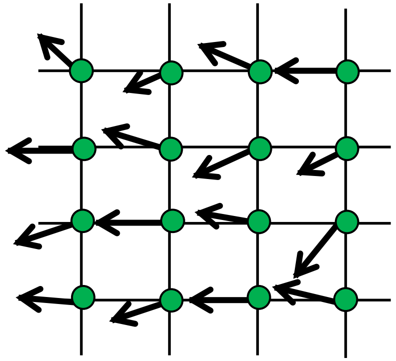
Paramagnetic Substance	χ	Diamagnetic Substance	χ
Aluminum	2.3×10^{-5}	Bismuth	-1.66×10^{-5}
Calcium	1.9×10^{-5}	Copper	-9.8×10^{-6}
Chromium	2.7×10^{-4}	Diamond	-2.2×10^{-5}
Lithium	2.1×10^{-5}	Gold	-3.6×10^{-5}
Magnesium	1.2×10^{-5}	Lead	-1.7×10^{-5}
Niobium	2.6×10^{-4}	Mercury	-2.9×10^{-5}
Oxygen	2.1×10^{-6}	Nitrogen	-5.0×10^{-9}
Platinum	2.9×10^{-4}	Silver	-2.6×10^{-5}
Tungsten	6.8×10^{-5}	Silicon	-4.2×10^{-6}

Values at 300 K

Diamagnetism and paramagnetism

$\boxed{\mathbf{B} = 0}$ $\boxed{\mathbf{B} \neq 0}$



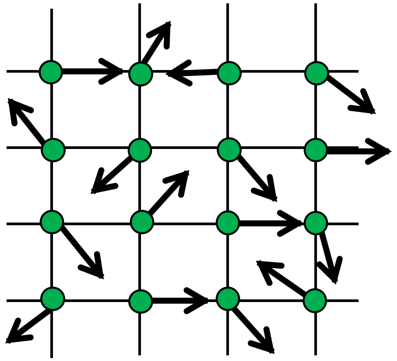


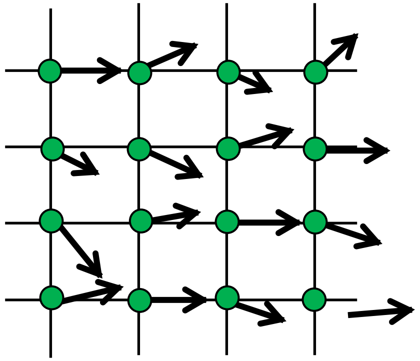
$\mathbf{M} = 0$ $\mathbf{M} \neq 0$

Diamagnetic atom

Atom without «permanent» magnetic dipole

Atom with «induced» magnetic dipole





$\mathbf{M} = 0$ $\mathbf{M} \neq 0$

Paramagnetic atom

Atom with «permanent» magnetic dipole

Diamagnetism

«Induced» magnetic diapoles not coupled

Atom without «permanent» magnetic dipole

Atom with «induced» magnetic dipole

$\mathbf{B} = 0 \;\Rightarrow\;$ $\mathbf{m}_i = 0, \mathbf{M} = 0$

$\mathbf{B} \neq 0 \Rightarrow$ the magnetic field induces a magnetic dipole in the atomes,

by "distorsion" (Lorentz force) of the mouvement of the electrons in the atomic orbitals.

This results in a magnetization $\mathbf{M}$ antiparallel to $\mathbf{B}$:

$\mathbf{M} // -\mathbf{B}$, $\mathbf{M} \equiv \chi_m \mathbf{H}$ with $\chi < 0$

It can be shown that, for a diamagnetic material:

$$\mathbf{M} = -\frac{ne^2\mu_0}{6m_e}\left(\sum_i r_i^2\right)_{moy}\mathbf{H} \qquad \Rightarrow \qquad \chi_m = -\frac{ne^2\mu_0}{6m_e}\left(\sum_i r_i^2\right)_{moy}$$

n : density of atoms [m^{-3}]

r_i : distance of the i electron from the nucleus [m]

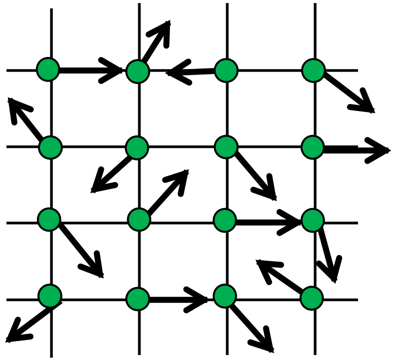
m_e : electron mass [kg]

e : electron charge [C]

Paramagnetism

«Permanent» magnetic diapoles not coupled

$\mathbf{B} = 0$

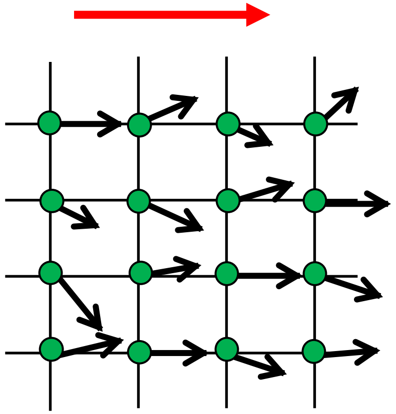


"competition" between

thermal energy (kT)

and magnetic energy (mB)

$\mathbf{B} \neq 0$



 Atom with «permanent» magnetic dipole

$\mathbf{B} = 0 \;\Rightarrow\; \mathbf{m}_i \neq 0, \mathbf{M} = 0$

$\mathbf{B} \neq 0 \;\Rightarrow\;$ The field $\mathbf{B}$ produces a torque $\mathbf{m}_i \times \mathbf{B}$ which tend to align the magnetic dipoles along the direction of the field $\mathbf{B}$ (the system has the tendency to minimise the magnetic potential energy $U_B = -\sum_i \mathbf{m}_i \cdot \mathbf{B}$),

The random atomic/molecular mouvements due to the temperature tends to cancel the aligment due to the field $\mathbf{B}$ (the thermal energy $U_{th} \cong kT$ tends to orient randomnly the magnetic moments).

The final result of this "competition" is:

$\mathbf{M} // \mathbf{B}$, $\mathbf{M} \equiv \chi_m \mathbf{H}$ with $\chi_m > 0$

It can be shown that, for a paramagnetic material:

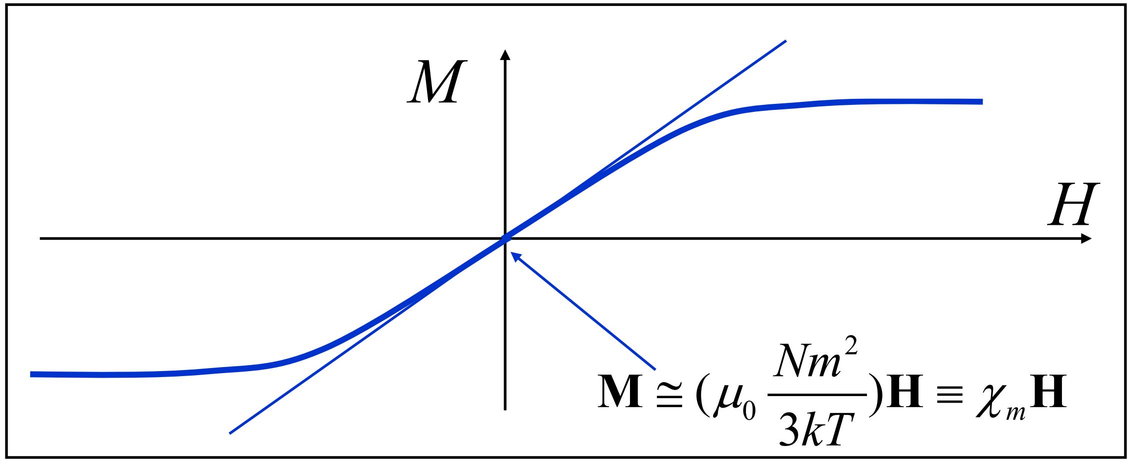
$$\mathbf{M} = Nm\left[\coth\left(\frac{\mu_0 mH}{kT}\right) - \frac{kT}{\mu_0 mH}\right]\frac{\mathbf{H}}{|\mathbf{H}|}$$

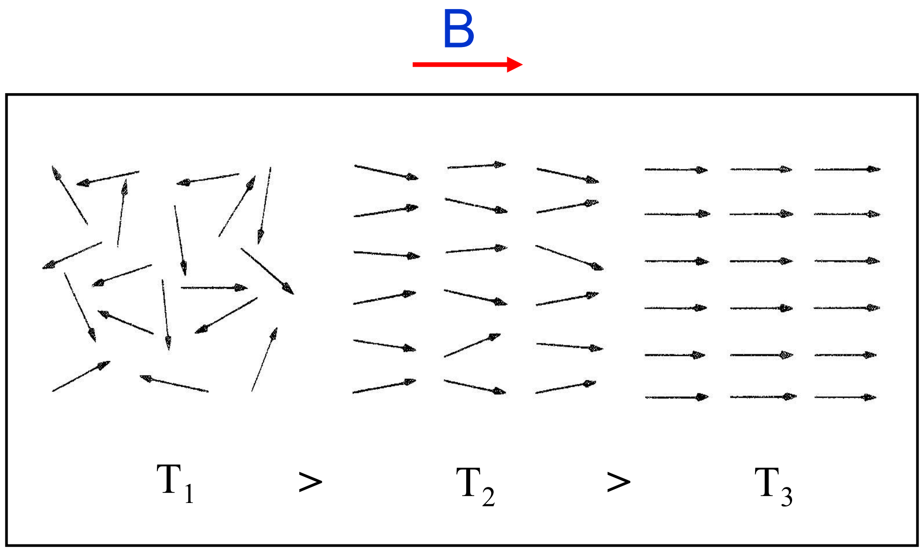
$$\text{Pour } \frac{\mu_0 mH}{kT} << 1 \Rightarrow \mathbf{M} \cong \frac{Nm^2\mu_0\mathbf{H}}{3kT} \Rightarrow \chi_m = \frac{Nm^2\mu_0}{3kT}$$

$$\frac{\mu_0 mH}{kT} >> 1 \Rightarrow \mathbf{M} \cong Nm$$

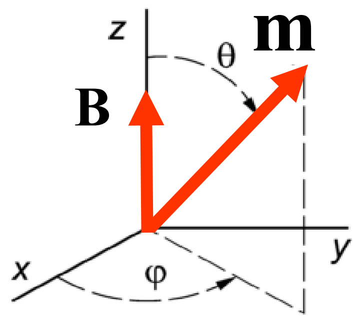
Pour $x<<1$: $\coth(x) - \frac{1}{x} \cong \frac{1}{3}x$

Loi de Curie





Some more details about the paramagnetism:



Energy of a magnetic moment **m** in a magnetic field **B** :

$$E = -\mathbf{m}\cdot\mathbf{B} \cong -\mathbf{m}\cdot\mu_0\mathbf{H} = -m\mu_0 H\cos\theta$$

H is the externally applied field (i.e., negligible field produced by neighbours magnetic moment, non-interacting dipoles)

"Randomization" of the magnetic moment orientation due to temperature:

$$M_x = M_y = 0$$

N is the density of magnetic moments (magnetic moments per unit of volume) [m^{-3}]

$$M_z = N\langle m_z\rangle = Nm\langle\cos\theta\rangle = Nm\frac{\int_0^{2\pi} d\varphi\int_0^{\pi}\sin\theta\cos\theta\exp\left(\frac{m\mu_0 H\cos\theta}{kT}\right)d\theta}{\int_0^{2\pi} d\varphi\int_0^{\pi}\sin\theta\exp\left(\frac{m\mu_0 H\cos\theta}{kT}\right)d\theta} =$$

$$= Nm\left(\coth\left(\frac{m\mu_0 H}{kT}\right) - \frac{kT}{m\mu_0 H}\right) = NmL\left(\frac{m\mu_0 H}{kT}\right)$$

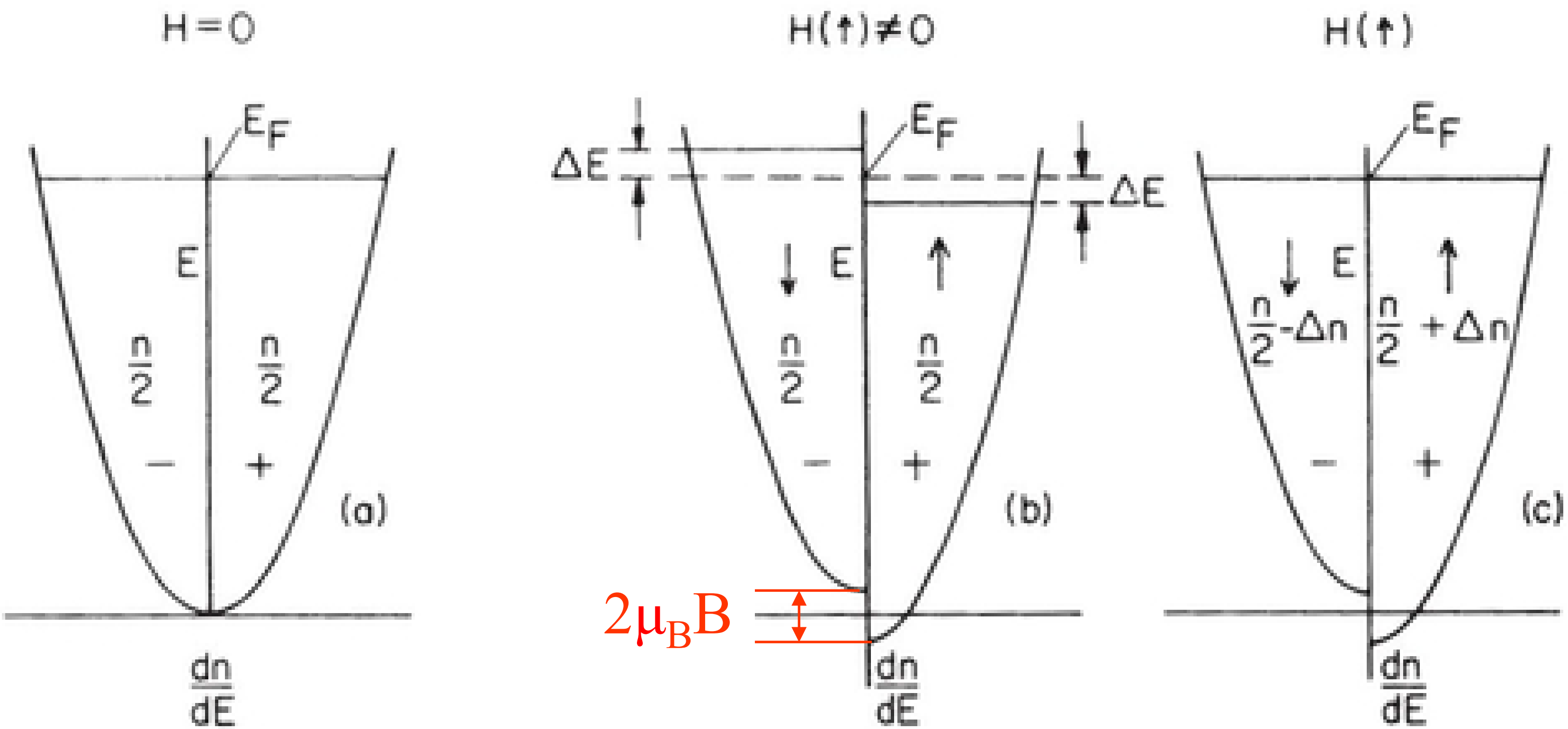
$$L(x) = \coth(x) - \frac{1}{x}$$

Langevin function

Pauli paramagnetism in metals

- Paramagnetism of free electron spins in a metal
- Almost temperature independent

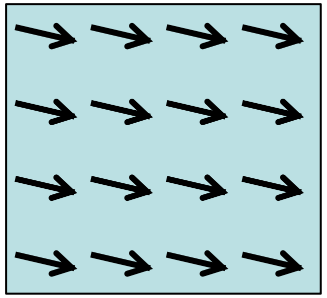
$$\chi_{Pauli} = \frac{3N\mu_0\,\mu_B^2}{2E_F}$$



Types of «ferromagnetism»

Magnetic domain

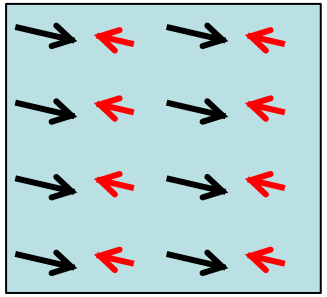
Ferromagnetism



The magnetic moments are parallel and have the sample amplitude

$\boxed{\mathbf{M} \neq 0}$

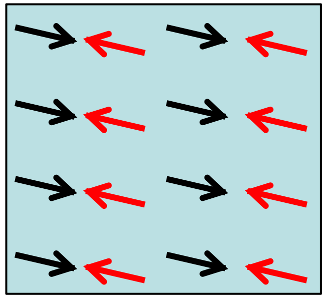
Ferrimagnetism



The magnetic moments are antiparallel and have different amplitude

$\boxed{\mathbf{M} \neq 0}$

Antiferromagnetism

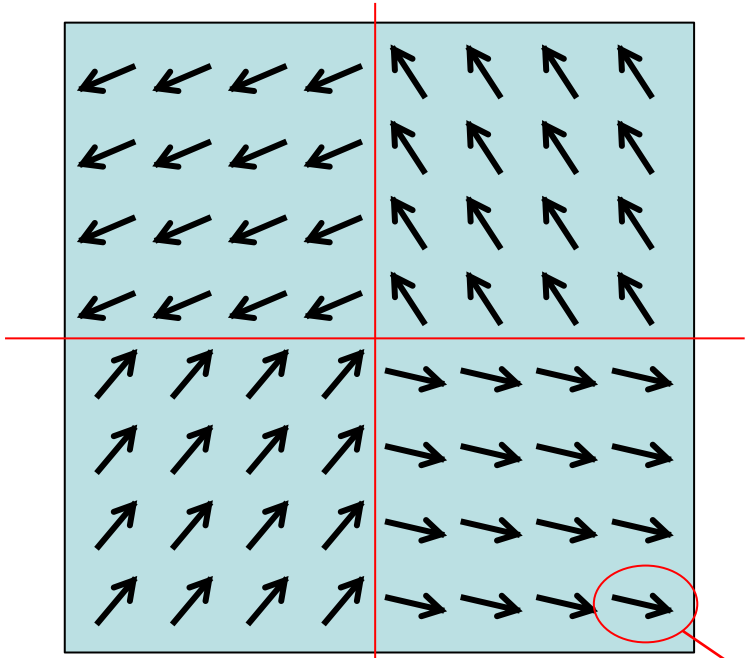


The magnetic moments are antiparallel and the same amplitude

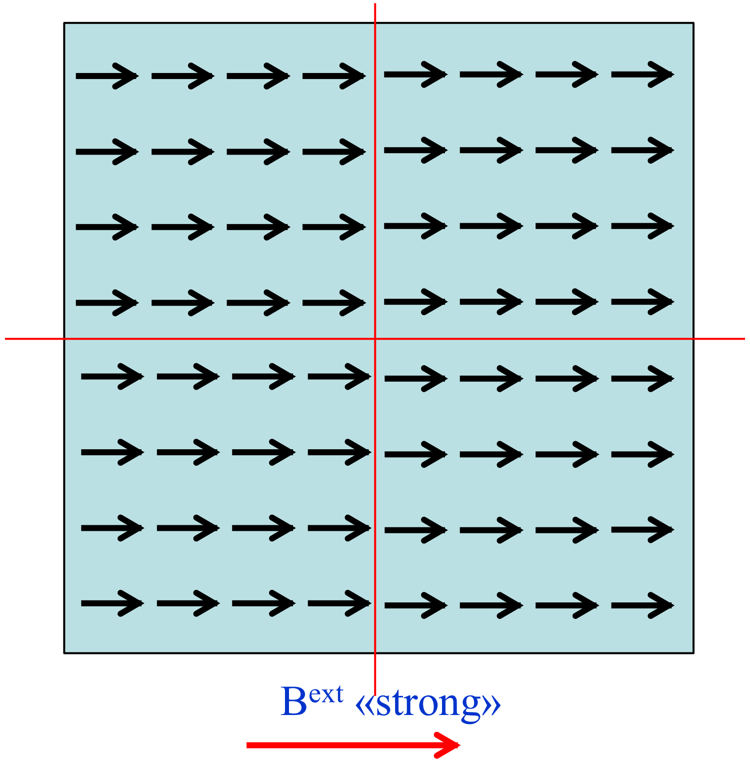
$\boxed{\mathbf{M} = 0}$

Ferromagnetism

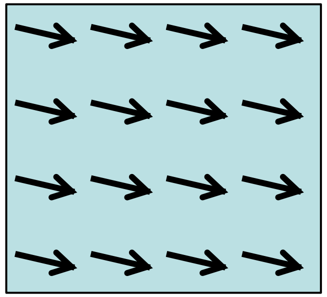
«Permanent» magnetic dipoles coupled



$B^{ext} = 0$



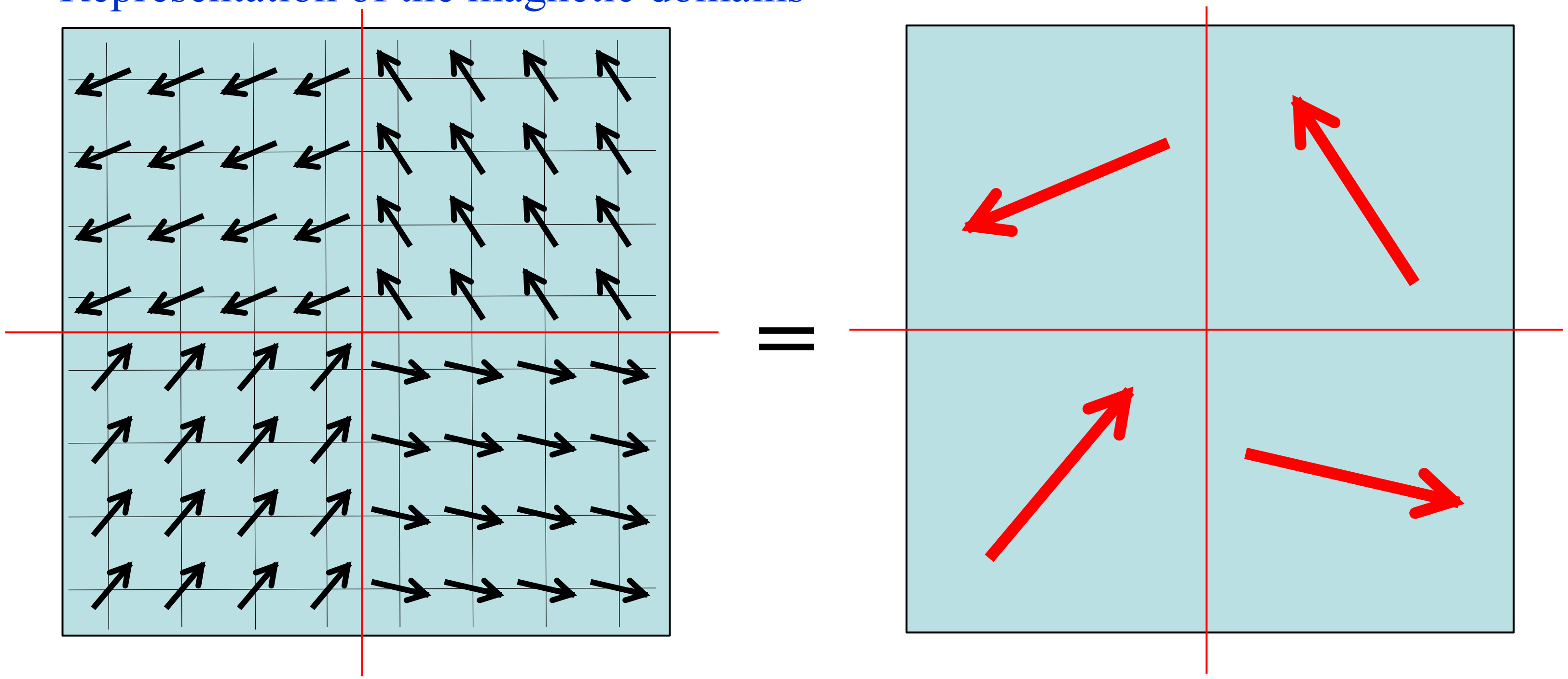
B^{ext} «strong»

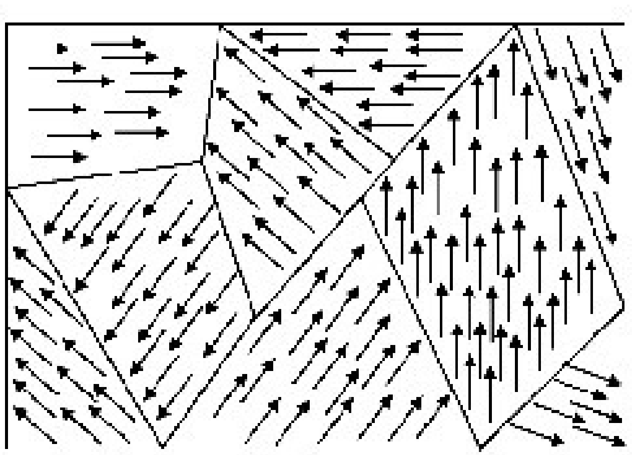


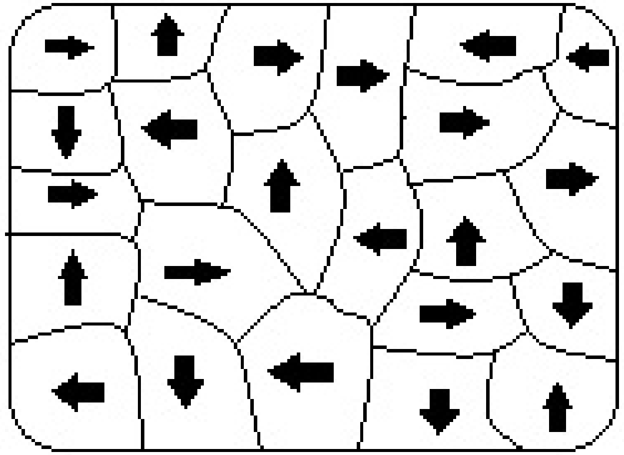
Magnetic domain:
Large number of atomes
with aligned magnetic moments.

«Magnetic» atom:
Atom with «permanent» magnetic dipole

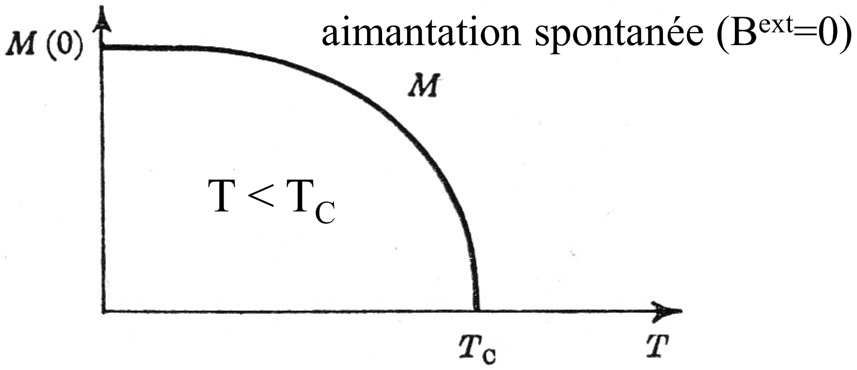
Representation of the magnetic domains



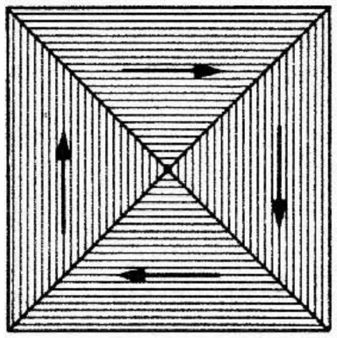




Magnetization in the magnetic domains vs **temperature**:

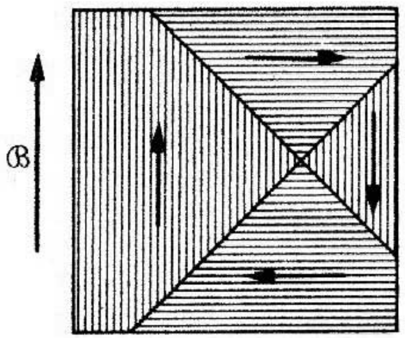


Magnetization in the magnetic domains vs **magnetic field**:



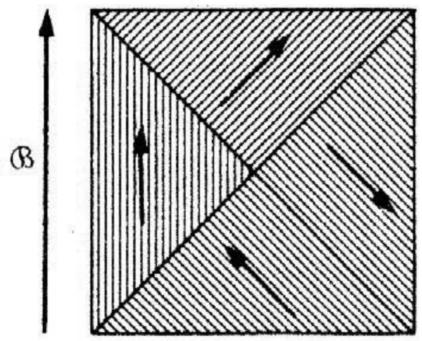
$B^{ext} = 0$

Magnetization by
growth of the domains



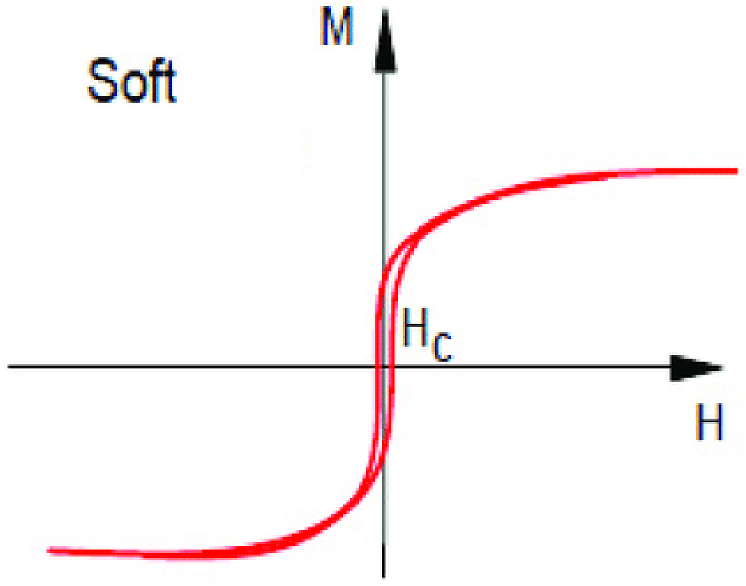
$B^{ext} > 0$

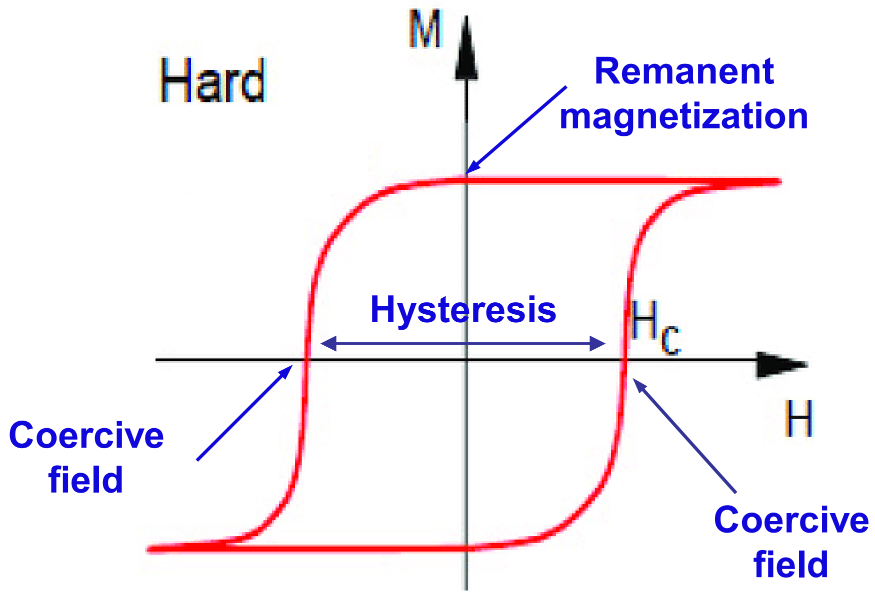
Magnetization by
orientation of the domains



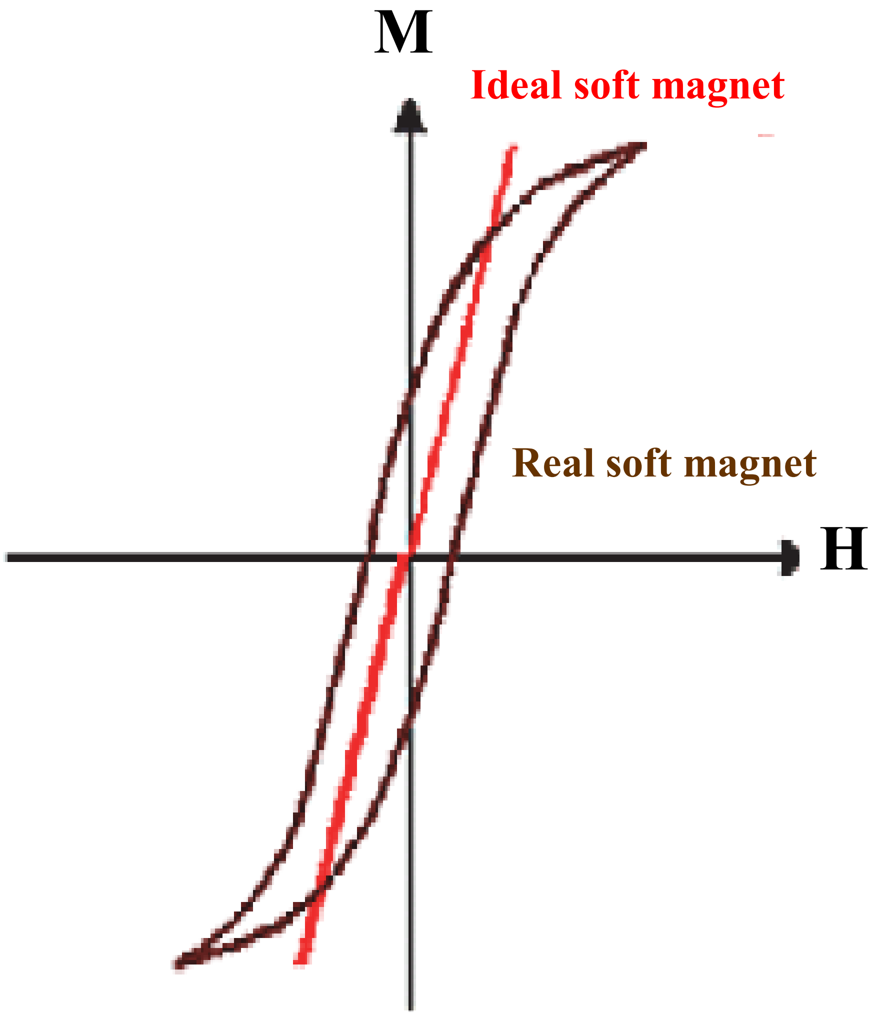
$B^{ext} > 0$

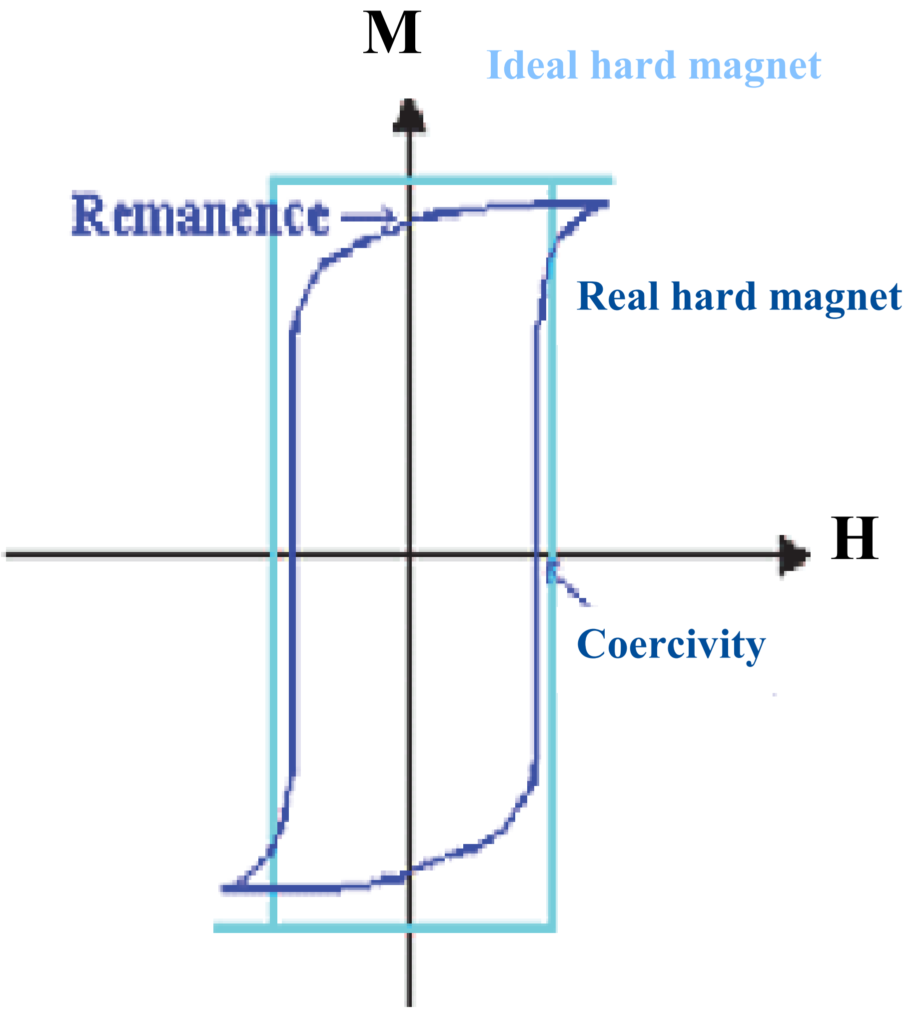
Soft and hard ferromagnetic materials



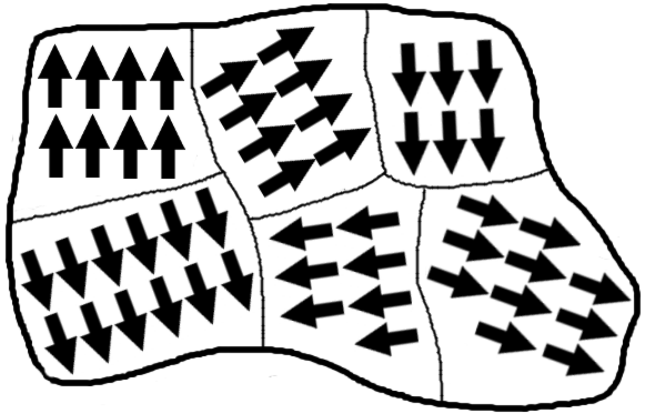


Remanent magnetization:	small		Remanent magnetization:	large
Hysteresis:	small		Hysteresis:	large
Coercive field:	small		Coercive field:	large


M
Ideal soft magnet
Real soft magnet
H


M
Ideal hard magnet
Remanence
Real hard magnet
H
Coercivity

Ferromagnetism: spontaneous magnetization



What is the *spontaneous magnetization* ?

Non zero magnetization that exists in individual regions (or domains) even when no external magnetic field is applied.

Why do we have *spontaneous magnetization* ?

The dipolar field created by the magnetic moment of a single atom is too weak to generate spontaneous magnetization at room temperature. This field is in the order of 1 T at a distance of 0.15 nm, i.e., at average distance between atoms in the solid. This would generate a significant magnetic order only below 1 K.

A purely quantum mechanical phenomenon is responsible of an *equivalent* magnetic field in the order of 1000 T. Weiss called it *molecular field*. Later it was called *exchange interaction.*

Ferromagnetism: Weiss theory

To explain **spontaneous magnetization,** Weiss introduced the influence of neighbours atoms as a <u>«molecular field», later called «exchange field».</u>

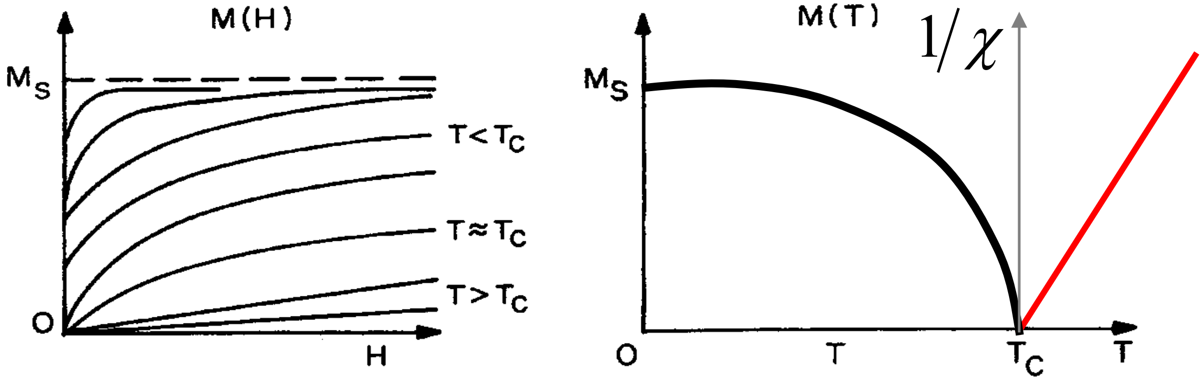
$$E = -\mathbf{m}\cdot\mathbf{B} \cong -\mathbf{m}\cdot\mu_0(\mathbf{H}+\lambda\mathbf{M})$$

$\lambda\mathbf{M}$: molecular field

$$M = NmL\left(\frac{\mu_0 m(H+\lambda M)}{kT}\right)$$

Langevin with «stronger» magnetic field

Curie temperature $T_c = \lambda\,\mu_0\,\dfrac{Nm^2}{3\,k}$



Spontaneous magnetization below Curie temperature

Paramagnetic above Curie temperature

$$\chi_{T>T_c} = \mu_0\,\frac{Nm^2}{3k(T-T_c)}$$

Ferromagnetism: Stoner-Wohlfarth theory

To explain the **hysteresis,** Stoner-Wohlfarth introduced the anisotropy energy with an easy axis

Zeeman (field) energy:

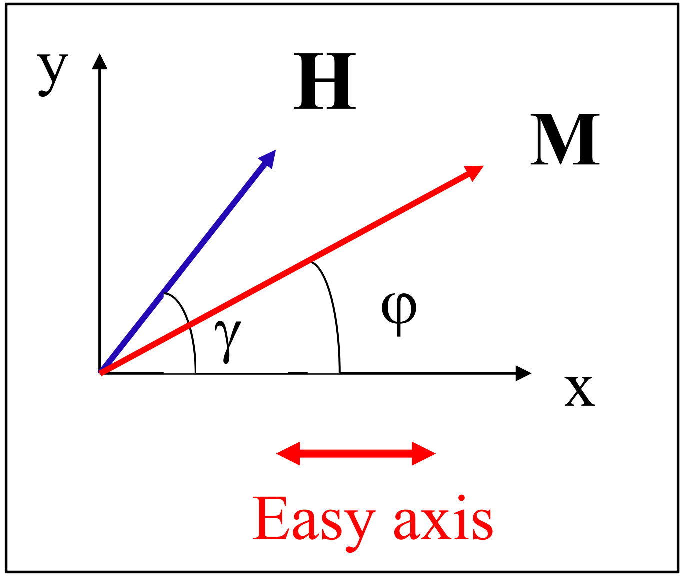
$$\boxed{E_z = -HM\cos(\gamma - \varphi)}$$

H : externally applied field

H_a : anisotropy field

Anisotropy energy:

$$\boxed{E_a = \frac{1}{2} H_a M \sin^2(\varphi)}$$

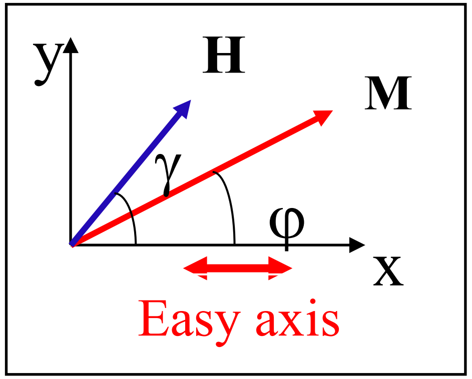


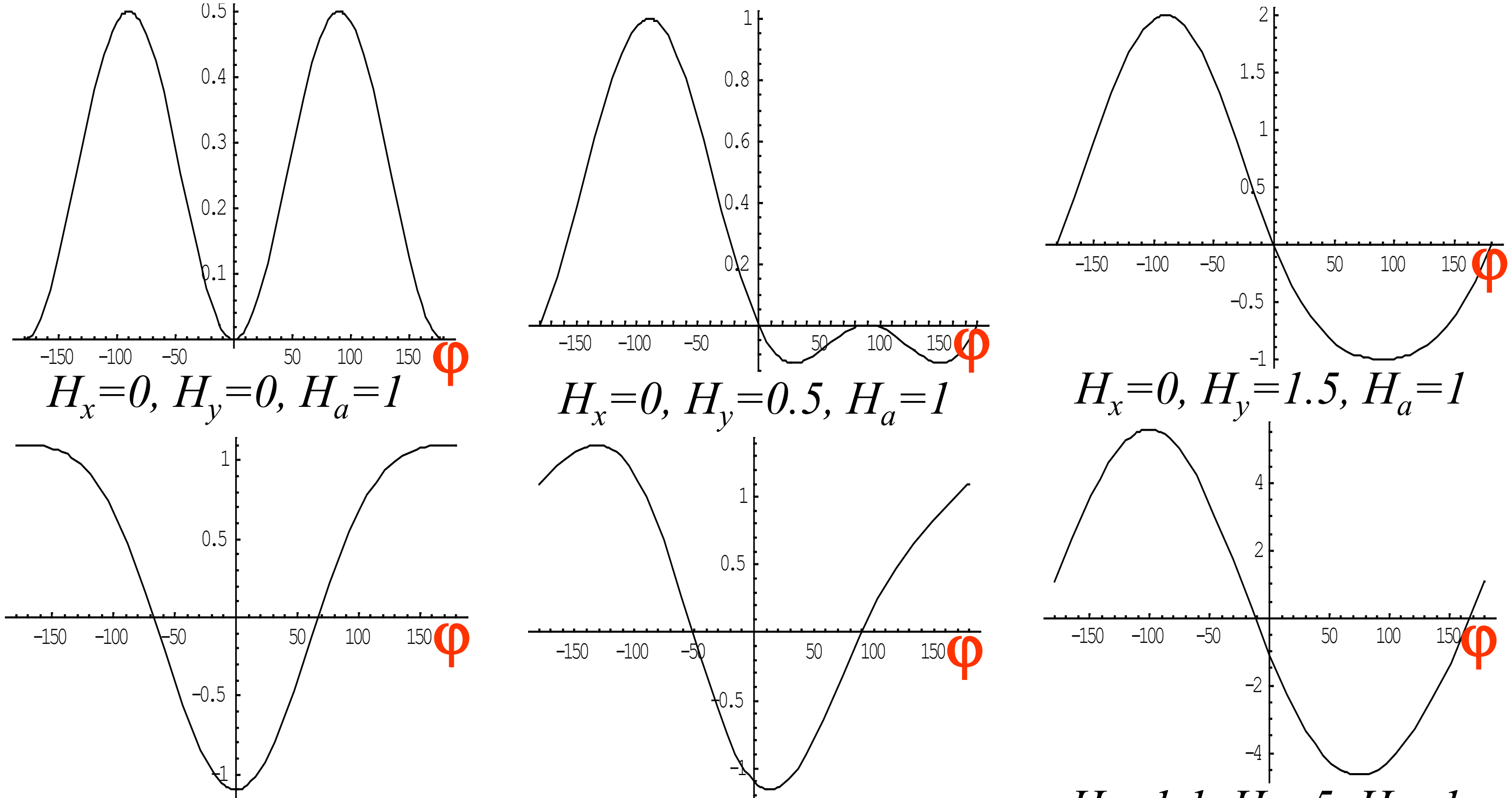
Energy minima: → determination of φ

determination of $\mathbf{M}(\mathbf{H})$

Energy vs φ
(minimum energy → stable φ)

$$E_{total} = \frac{1}{2} H_a M \sin^2(\varphi) - HM\cos(\gamma - \varphi)$$





H_x=0, H_y=0, H_a=1

H_x=0, H_y=0.5, H_a=1

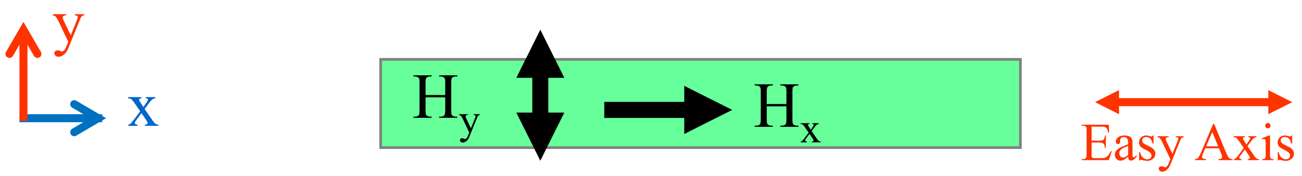
H_x=0, H_y=1.5, H_a=1

H_x=1.1, H_y=0, H_a=1

H_x=1.1, H_y=0.5, H_a=1

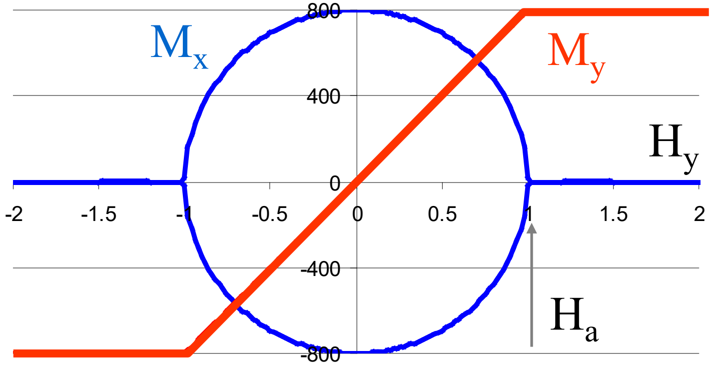
H_x=1.1, H_y=5, H_a=1

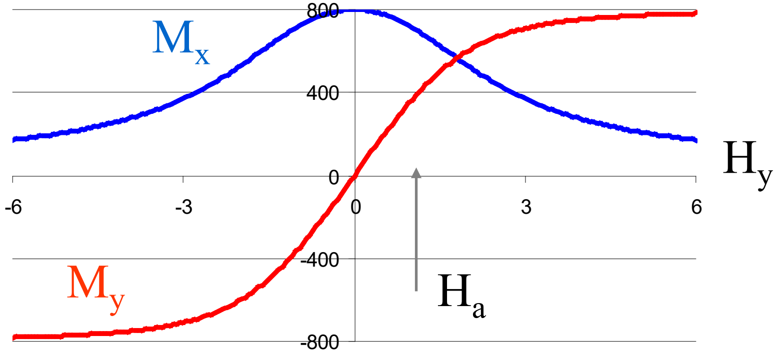
Field perpendicular to the easy axis



a) Perpendicular field H_y with H_x=0, H_a=1

b) Perpendicular field H_y with H_x=1.1, H_a=1

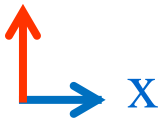


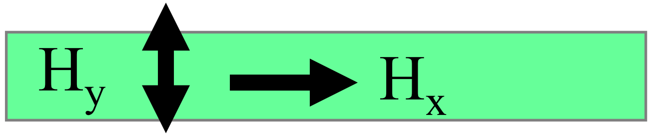


→Hysteresis

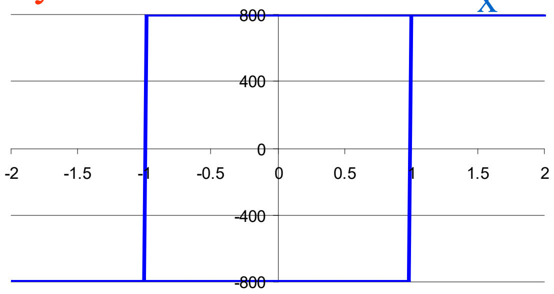
→No hysteresis

Field parallel to easy axis



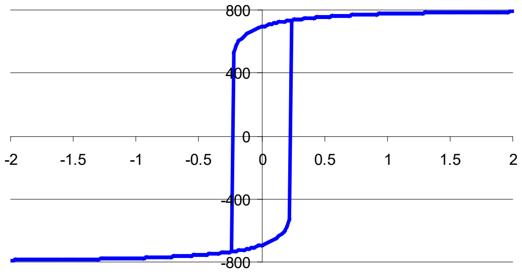


H_y=0 M_x



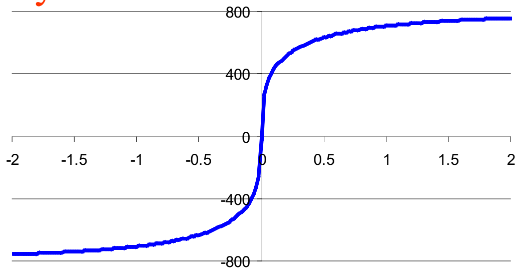
H_x

H_y=0.5 M_x



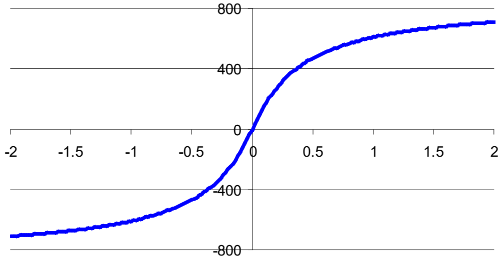
H_x

H_y=1 M_x



H_x

H_y=1.5 M_x



H_x

H_a=1

Antiferromagnetism

Two kinds of atoms with the same but antiparallel magnetization

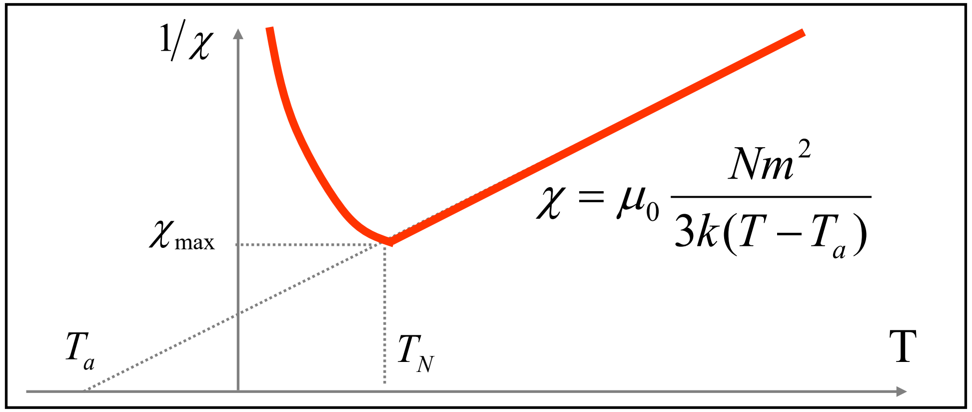
$$\boxed{E_1 = -\mathbf{m}\cdot\mu_0(\mathbf{H} + \lambda_1\mathbf{M}_1 - \lambda_2\mathbf{M}_2)}$$

$$\boxed{E_2 = -\mathbf{m}\cdot\mu_0(\mathbf{H} + \lambda_1\mathbf{M}_2 - \lambda_2\mathbf{M}_1)}$$

$$\boxed{M_1 = NmL(\frac{E_1}{kT})}$$

$$\boxed{M_2 = NmL(\frac{E_2}{kT})}$$

$$\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2$$



Néel Temperature

$$T_N = (\lambda_1 + \lambda_2)\,\mu_0\frac{Nm^2}{3k}$$

$$T_a = (\lambda_1 - \lambda_2)\,\mu_0\frac{Nm^2}{3k}$$

$$\chi_{\max} = \lambda_2$$

No spontaneous magnetization

Ferrimagnetism

Two kinds of atoms with different and antiparallel magnetization

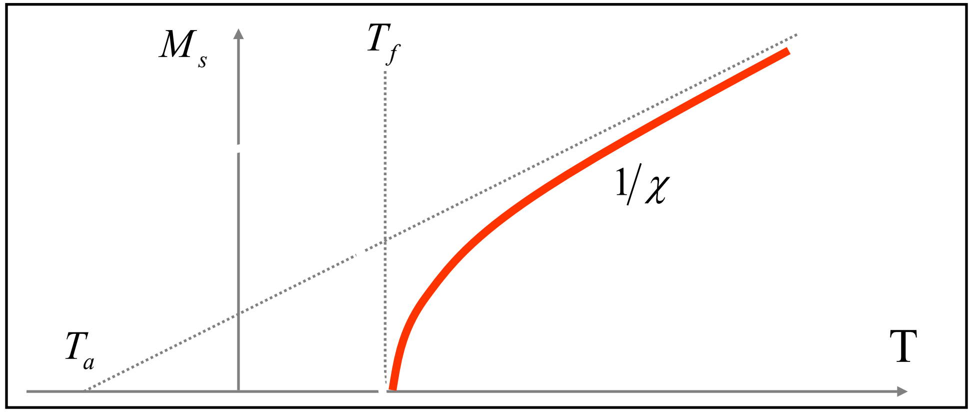
$$\boxed{E_1 = -\mathbf{m}_1 \cdot \mu_0 (\mathbf{H} + \lambda_{11}\mathbf{M}_1 - \lambda_{12}\mathbf{M}_2)}$$

$$\boxed{E_2 = -\mathbf{m}_2 \cdot \mu_0 (\mathbf{H} + \lambda_{22}\mathbf{M}_2 - \lambda_{21}\mathbf{M}_1)}$$

$$\boxed{M_1 = N_1 m_1 L(\frac{E_1}{kT})}$$

$$\boxed{M_2 = N_2 m_2 L(\frac{E_2}{kT})}$$

$$\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2$$



Below T_f

Spontaneous magnetization

Above T_f

Paramagnetic like behavior

Forces and torques

The equations of **forces** and **torques**
are all derived from the Lorentz force
(the electromagnetic force)

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v}\times\mathbf{B} \quad \Rightarrow \quad \begin{cases} \mathbf{F} = \int_V \left(\rho\mathbf{E} + \mathbf{J}\times\mathbf{B}\right) dV & \mathbf{N} = \int_V \left(\mathbf{r}\times\rho\mathbf{E} + \mathbf{r}\times(\mathbf{J}\times\mathbf{B})\right) dV \\ \mathbf{F} = \nabla(\mathbf{p}\cdot\mathbf{E}) & \mathbf{N} = \mathbf{p}\times\mathbf{E} \\ \mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B}) & \mathbf{N} = \mathbf{m}\times\mathbf{B} \end{cases}$$

Note: (see Z 96, Z 373, P 478, P 538)

The "exact" expression (i.e., always valid) of the force on an electric dipole is $\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E}$

but for $\nabla\times\mathbf{E} = 0$ and $\mathbf{p}$ independent on the position: $\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E} = \nabla(\mathbf{p}\cdot\mathbf{E})$

The "exact" expression (i.e., always valid) of the force on a magnetic dipole is $\mathbf{F} = m_k \nabla B_k$

but for $\mathbf{m}$ independent on the position: $\mathbf{F} = m_k \nabla B_k = \nabla(\mathbf{m}\cdot\mathbf{B})$

for $\mathbf{m}$ independent on the position and $\nabla\times\mathbf{B} = 0$: $\mathbf{F} = m_k \nabla B_k = \nabla(\mathbf{m}\cdot\mathbf{B}) = (\mathbf{m}\cdot\nabla)\mathbf{B}$

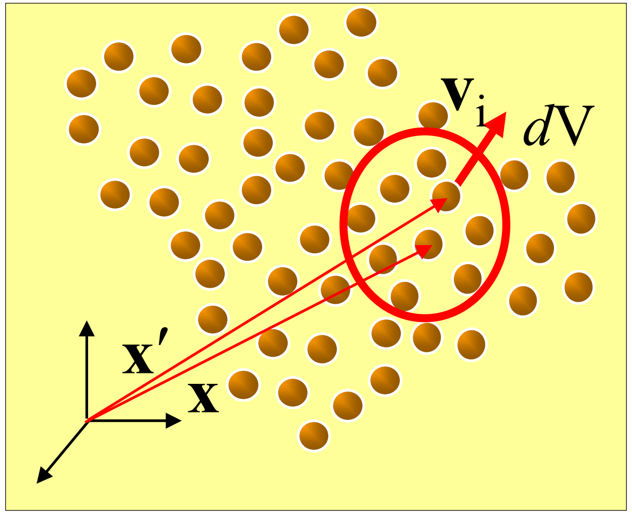
Force and torque on a distribution of currents in a magnetic field:

From **F**=q**v**x**B** on a single charge, the total force is:

$$\mathbf{F} = \sum_i q_i \mathbf{v}_i \times \mathbf{B}$$

Since the current density is defined as:

$$\mathbf{J}(\mathbf{x}) = \frac{1}{dV} \sum_i q_i \mathbf{v}_i$$

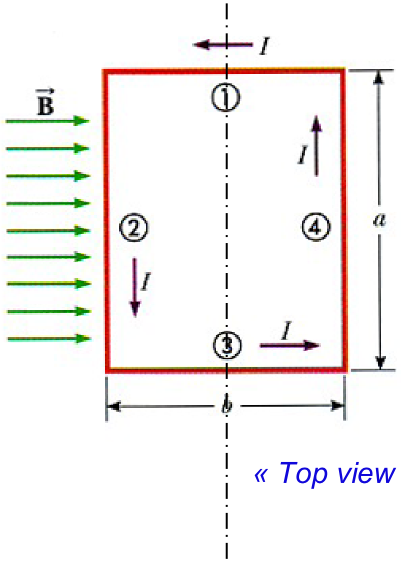


And, consequently, the total force is:

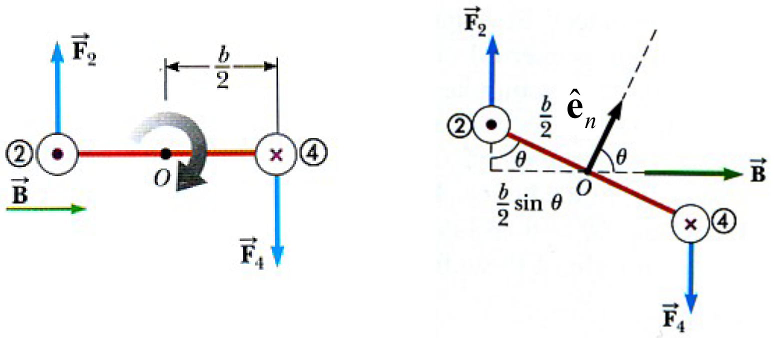
$$\mathbf{F} = \int_V \mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x}) d\mathbf{x}$$ Total force

$$\mathbf{N} = \int_V \mathbf{x} \times \left(\mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x}) \right) d\mathbf{x}$$ Total torque

Force and torque on a circuit in a uniform magnetic field:



« Top view »



« Lateral view »

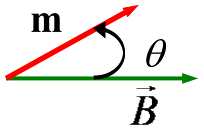
$$\mathbf{F} = I\mathbf{L}\times\mathbf{B} \quad\Rightarrow\quad F_2 = F_4 = IaB\ ;\quad \mathbf{F}_1 + \mathbf{F}_3 = 0$$

$$\boldsymbol{\tau} = \mathbf{r}\times\mathbf{F} \quad\Rightarrow\quad \tau = bIaB\sin\theta = ISB\sin\theta\ ;\quad \mathbf{S} = ab\,\hat{\mathbf{e}}_n = S\hat{\mathbf{e}}_n$$

$$\Rightarrow$$

$$\boldsymbol{\tau} = I\mathbf{S}\times\mathbf{B} = \mathbf{m}\times\mathbf{B}$$ Torque

$$\mathbf{m} \equiv I\mathbf{S}$$ «Magnetic dipole» of the circuit



$d\theta > 0 \rightarrow dW < 0$

$$dW = -\tau\, d\theta = -|\mathbf{m}|\, B\sin\theta\, d\theta = -dU_m$$

$$U_m = -|\mathbf{m}|\, B\cos\theta = -\mathbf{m}\cdot\mathbf{B}$$ Magnetic potential energy

Torque on a circular loop in a uniform magnetic field

$$d\mathbf{F} = Id\mathbf{l}\times\mathbf{B}$$

$$d\mathbf{N} = \mathbf{r}\times d\mathbf{F} = \mathbf{r}\times\left(Id\mathbf{l}\times\mathbf{B}\right)$$

donc si $\mathbf{r} = R\hat{\mathbf{r}}$ et $\mathbf{B} = B\hat{\mathbf{B}}$

$$\Rightarrow$$

$$d\mathbf{N} = IRB\hat{\mathbf{r}}\times\left(d\mathbf{l}\times\hat{\mathbf{B}}\right)$$

$$\Rightarrow$$

$$\mathbf{N} = \int_0^{2\pi} d\mathbf{N}d\varphi = \mathbf{m}\times\mathbf{B}$$

$$\hat{\mathbf{r}} = \cos\varphi\hat{\mathbf{x}} + \sin\varphi\hat{\mathbf{y}}$$

$$d\mathbf{l} = -Rd\varphi\sin\varphi\hat{\mathbf{x}} + Rd\varphi\cos\varphi\hat{\mathbf{y}}$$

$$d\hat{\mathbf{l}} = -\sin\varphi\hat{\mathbf{x}} + \cos\varphi\hat{\mathbf{y}}$$

$$\hat{\mathbf{B}} = \sin\theta\cos\alpha\hat{\mathbf{x}} + \sin\theta\sin\alpha\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$

$$\Rightarrow$$

$$d\hat{\mathbf{l}}\times\hat{\mathbf{B}} = \cos\varphi\cos\theta\hat{\mathbf{x}} + \sin\varphi\cos\theta\hat{\mathbf{y}} + \left(-\sin\varphi\sin\theta\sin\alpha - \cos\alpha\sin\theta\cos\varphi\right)\hat{\mathbf{z}}$$

$$\Rightarrow$$

$$\begin{aligned}\hat{\mathbf{r}}\times(d\hat{\mathbf{l}}\times\hat{\mathbf{B}}) = {} & \sin\varphi\left(-\sin\varphi\sin\theta\sin\alpha - \cos\alpha\sin\theta\cos\varphi\right)\hat{\mathbf{x}} \\ & -\cos\varphi\left(-\sin\varphi\sin\theta\sin\alpha - \cos\alpha\sin\theta\cos\varphi\right)\hat{\mathbf{y}}\end{aligned}$$

$$\Rightarrow$$

$$d\mathbf{N} = IR^2B\hat{\mathbf{r}}\times(d\hat{\mathbf{l}}\times\hat{\mathbf{B}}) = IR^2Bd\varphi\begin{bmatrix}\sin\varphi\left(-\sin\varphi\sin\theta\sin\alpha - \cos\alpha\sin\theta\cos\varphi\right)\hat{\mathbf{x}} \\ -\cos\varphi\left(-\sin\varphi\sin\theta\sin\alpha - \cos\alpha\sin\theta\cos\varphi\right)\hat{\mathbf{y}}\end{bmatrix}$$

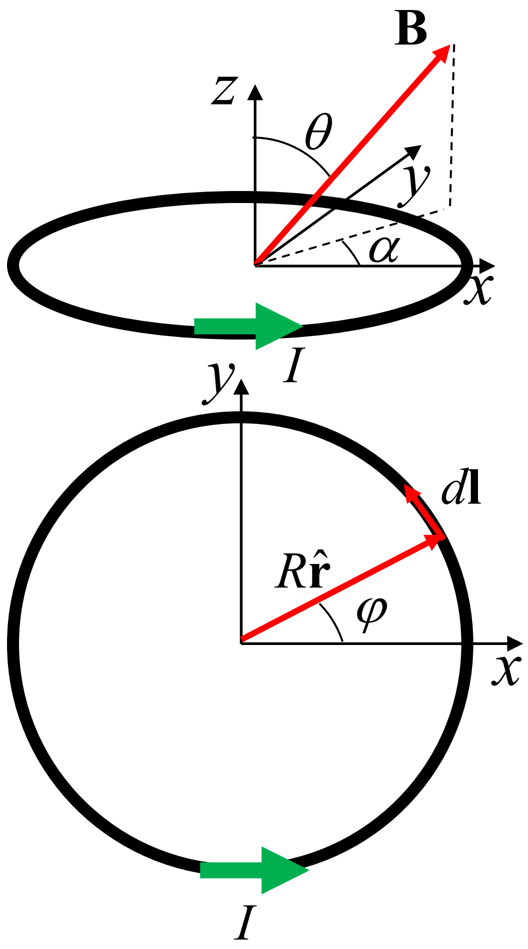
$$\Rightarrow \mathbf{N} = \int_0^{2\pi} d\mathbf{N}d\varphi = I\pi R^2B\left(-\sin\theta\sin\alpha\hat{\mathbf{x}} - \sin\theta\cos\alpha\hat{\mathbf{y}}\right) =$$

$$= mB\left(-\sin\theta\sin\alpha\hat{\mathbf{x}} - \sin\theta\cos\alpha\hat{\mathbf{y}}\right) = \mathbf{m}\times\mathbf{B}$$

$$\int_0^{2\pi}\sin^2\varphi d\varphi = \int_0^{2\pi}\cos^2\varphi d\varphi = \pi \qquad \int_0^{2\pi}\sin\varphi\cos\varphi d\varphi = 0$$

$$\mathbf{m} = m\hat{\mathbf{z}} = I\pi R^2\hat{\mathbf{z}}$$

$$\mathbf{B} = B\left(\sin\theta\cos\alpha\hat{\mathbf{x}} + \sin\theta\sin\alpha\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}\right)$$



$$\mathbf{m} = \frac{1}{2}\int_V \mathbf{r}\times\mathbf{J}(\mathbf{r})dV = \frac{1}{2}I\oint_C \mathbf{r}\times d\mathbf{l} = \frac{1}{2}I\hat{\mathbf{z}}\oint_C Rdl = \frac{1}{2}I\hat{\mathbf{z}}R\oint_C dl$$

$$= \frac{1}{2}IR2\pi R\hat{\mathbf{z}} = I\pi R^2\hat{\mathbf{z}}$$

Force and torque on a magnetic object in a magnetic field **B**:

If B is not varying too quickly over the volume of the object we can assume:

$$B_i(\mathbf{x}) = B_i(0) + \mathbf{x} \cdot \nabla B_i(0) \quad (i = x, y, z)$$

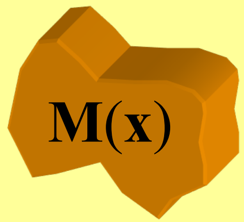
where **B**(**x**) is the externally applied field

With this assumption we can demonstrate that:

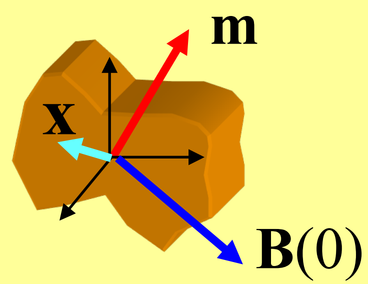
$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$ Force

$\mathbf{N} = \mathbf{m} \times \mathbf{B}$ Torque

$U = -\mathbf{m} \cdot \mathbf{B}$ Potential energy



$$\mathbf{m} = \int_V \mathbf{M}(\mathbf{x}) d\mathbf{x}$$

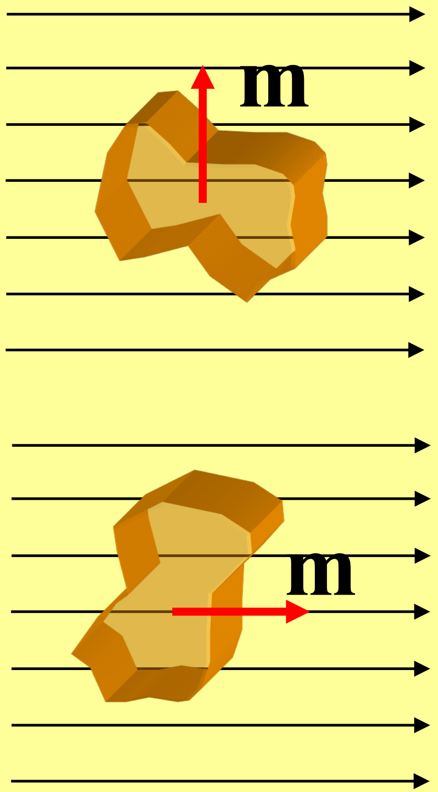


Problem:
Given **M**(**x**) and the ext. applied **B**(**x**), compute the **F** and **N** on the magnetic object

$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B}) \qquad \text{Force}$$

$$\mathbf{N} = \mathbf{m} \times \mathbf{B} \qquad \text{Torque}$$

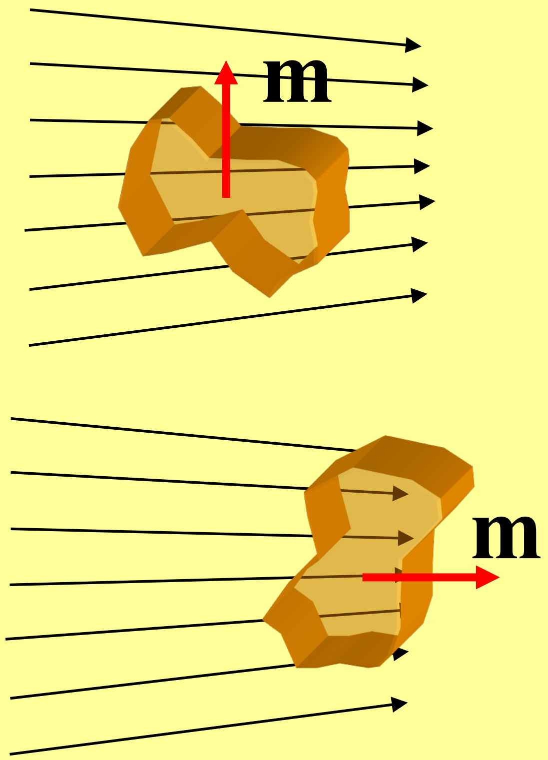
1. **Uniform** magnetic field



$\mathbf{F} = 0 \qquad (\textit{no translation})$

$\mathbf{N} \neq 0 \qquad (\textit{rotation})$

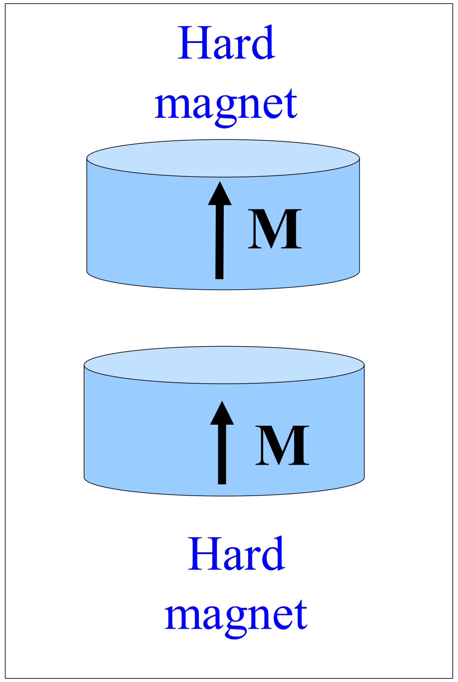
2. **Non-uniform** magnetic field



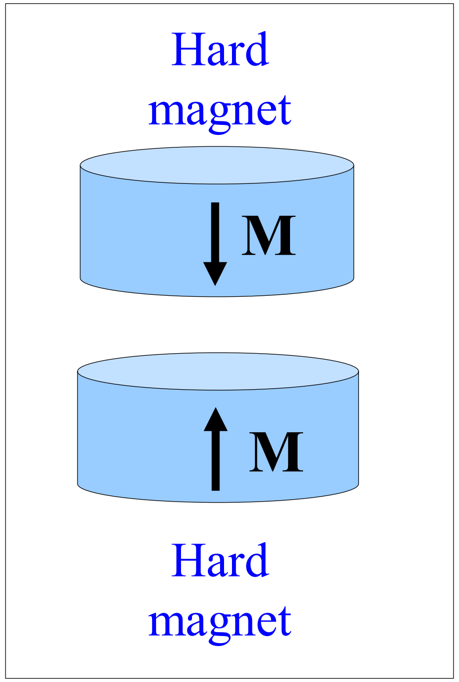
$\mathbf{F} \neq 0 \qquad (\textit{translation})$

$\mathbf{N} \neq 0 \qquad (\textit{rotation})$

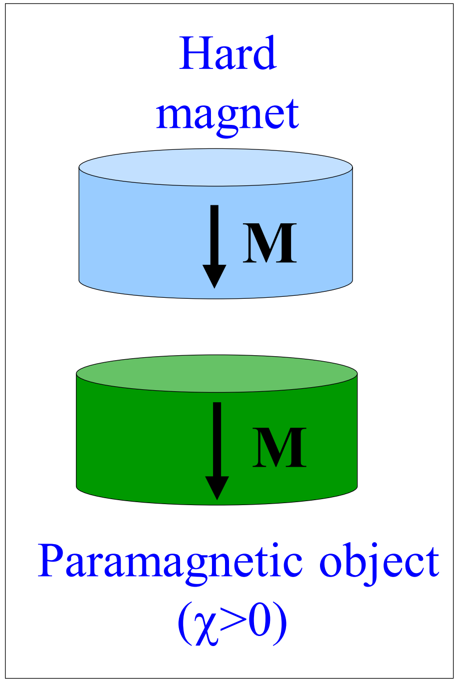
Magnetic levitation of bulk materials



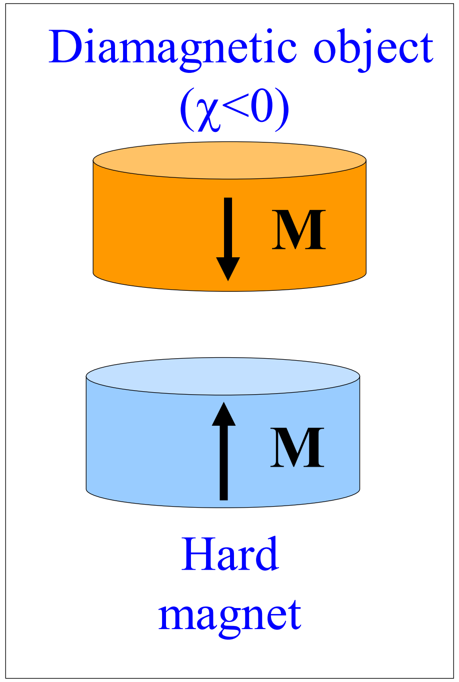
Non-equilibrium



«Almost stable» equilibrium
(rotation)

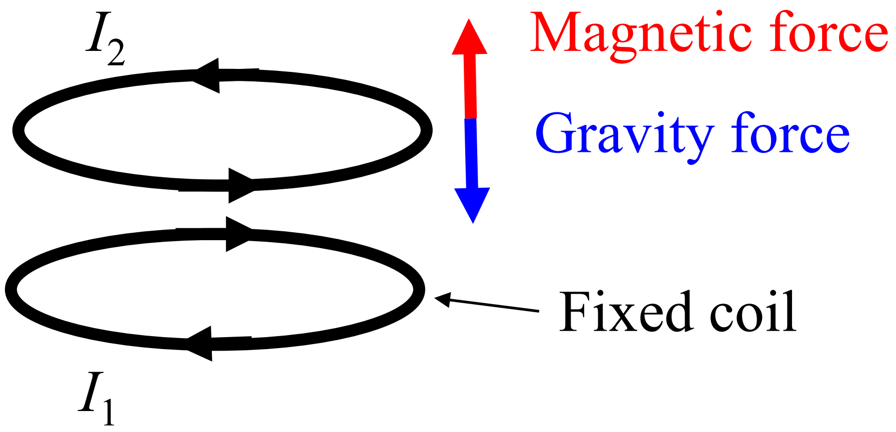


Non-equilibrium

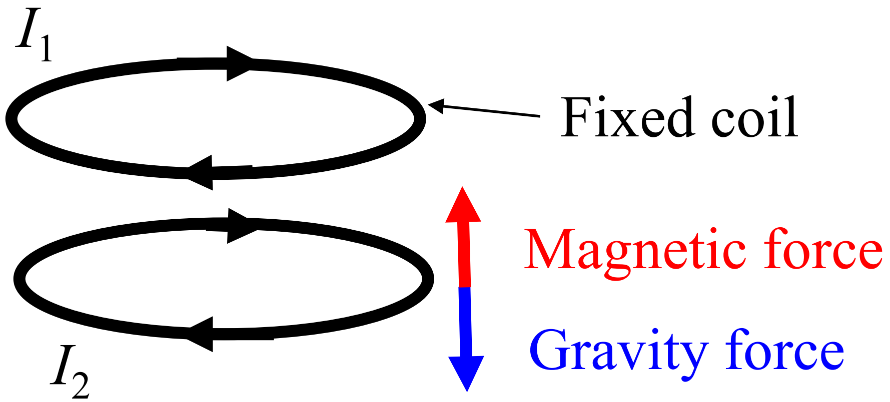


«Almost stable» equilibrium
(rotation)

Magnetic levitation of a loop with a current

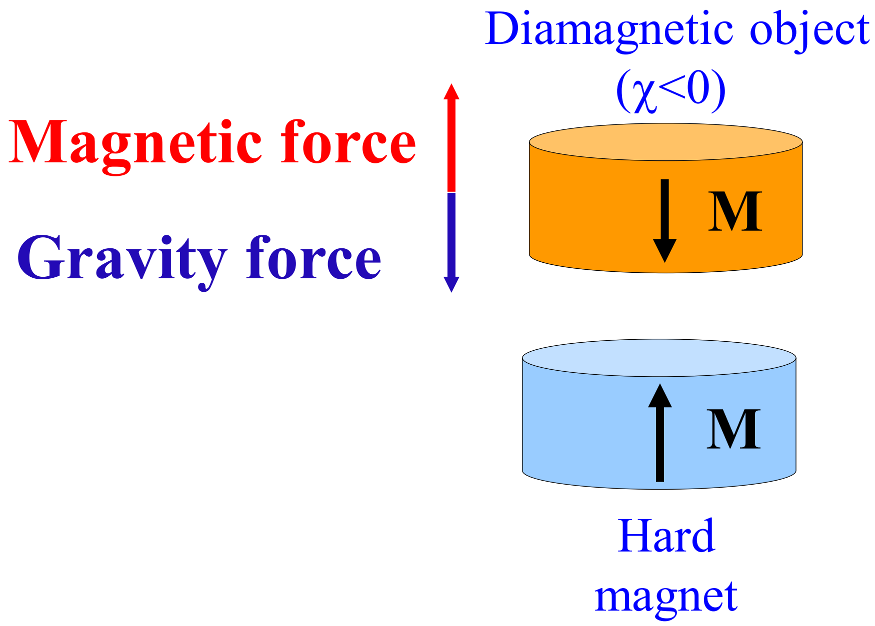


Stable equilibrium



Unstable equilibrium

Diamagnet object levitation



«Almost stable» equilibrium
(rotation)

Energy

$$E_{magnetic} = -\int_V \mathbf{M} \cdot \mathbf{B} dV$$
$$E_{gravity} = mgh$$

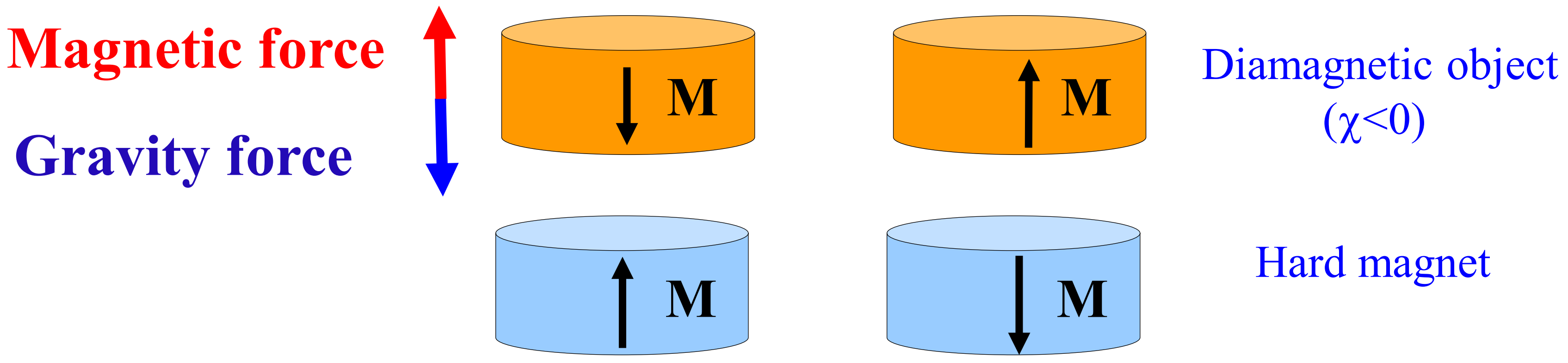
Force

$$F_{magnetic} = \nabla E_{magnetic} = ...$$
$$F_{gravity} = \nabla E_{gravity} = mg$$

Position

$$F_{magnetic} + F_{gravity} = 0$$

Diamagnetic object levitation



Energie potentielle

$$U_B \cong -\mathbf{m} \cdot \mathbf{B}$$

$$U_G \cong mgh$$

Force

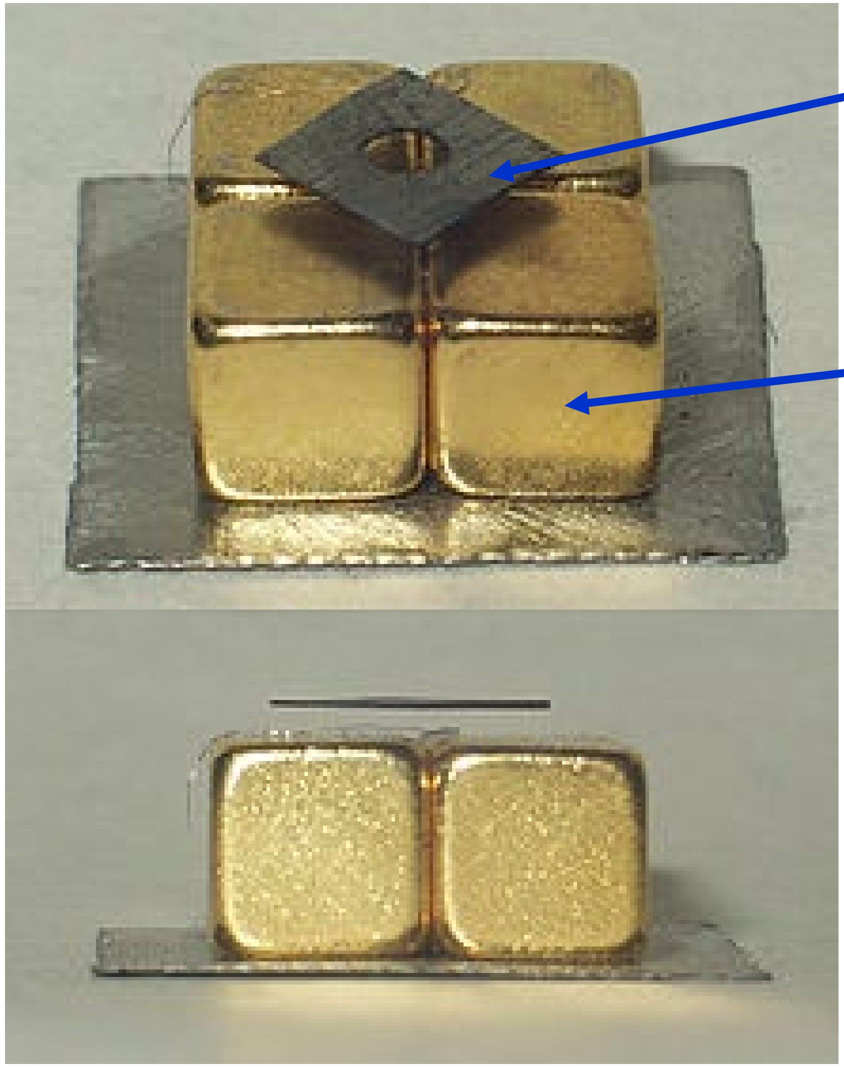
$$\mathbf{F}_B = -\nabla U_B \cong \nabla(\mathbf{m} \cdot \mathbf{B})$$

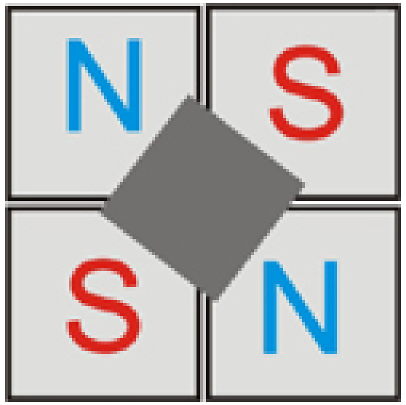
$$\mathbf{F}_G = -\nabla U_G = m\mathbf{g}$$

Position

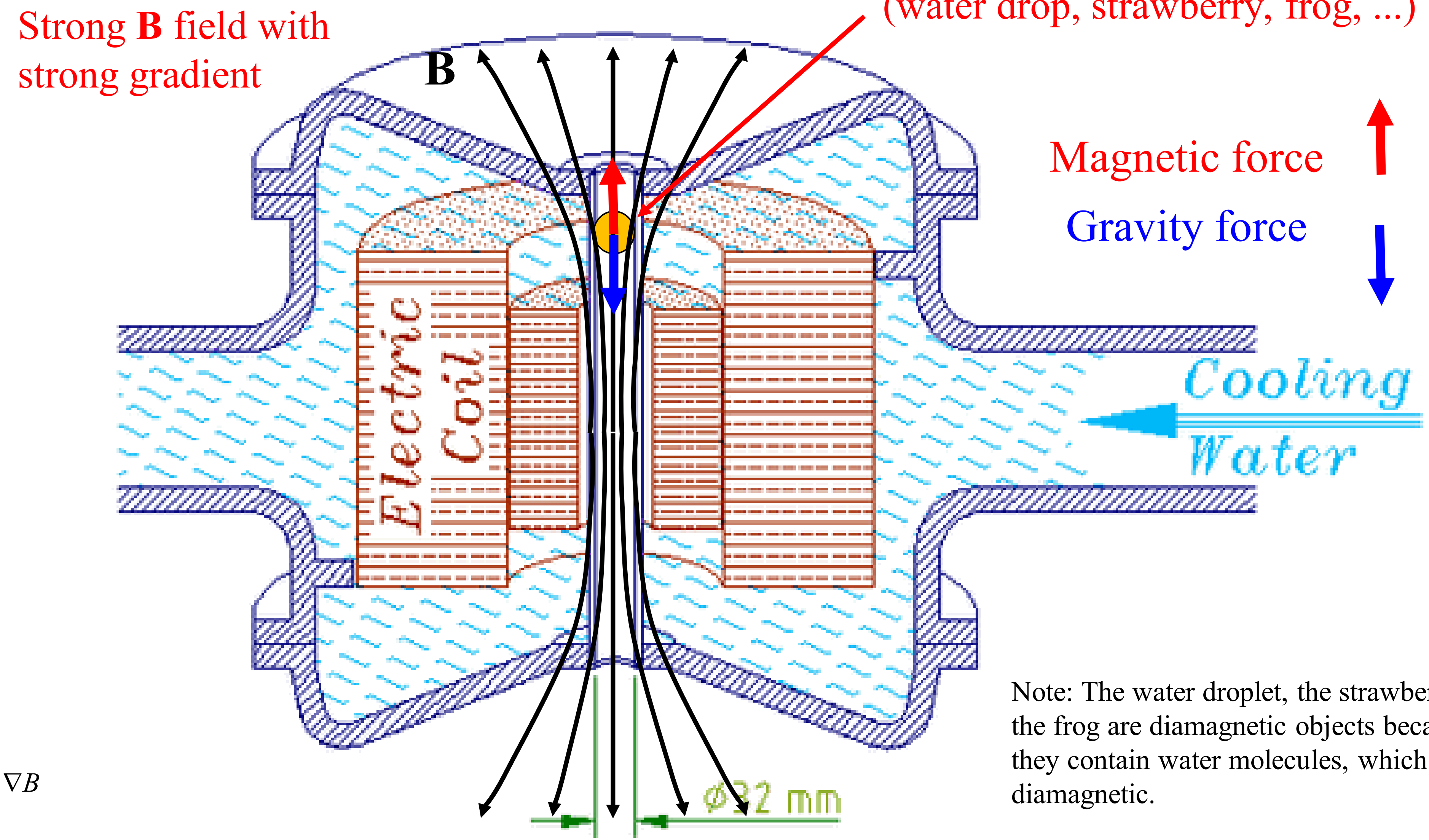
$$\mathbf{F}_B + \mathbf{F}_G = 0$$

1. Levitation with an array of permanent magnets





2. Levitation with a solenoidal coil



V : objet volume (m^3)

m : objet mass $\quad m = \rho V$ (kg)

ρ : objet density (kg/m^3)

$$\mathbf{F}_B = \nabla(\mathbf{m} \cdot \mathbf{B})$$

$$\mathbf{m} \cong \mathbf{M}V \cong \chi_m \mathbf{H} V \cong \chi_m \frac{\mathbf{B}}{\mu_0} V$$

$$\mathbf{F}_G = m\mathbf{g}$$

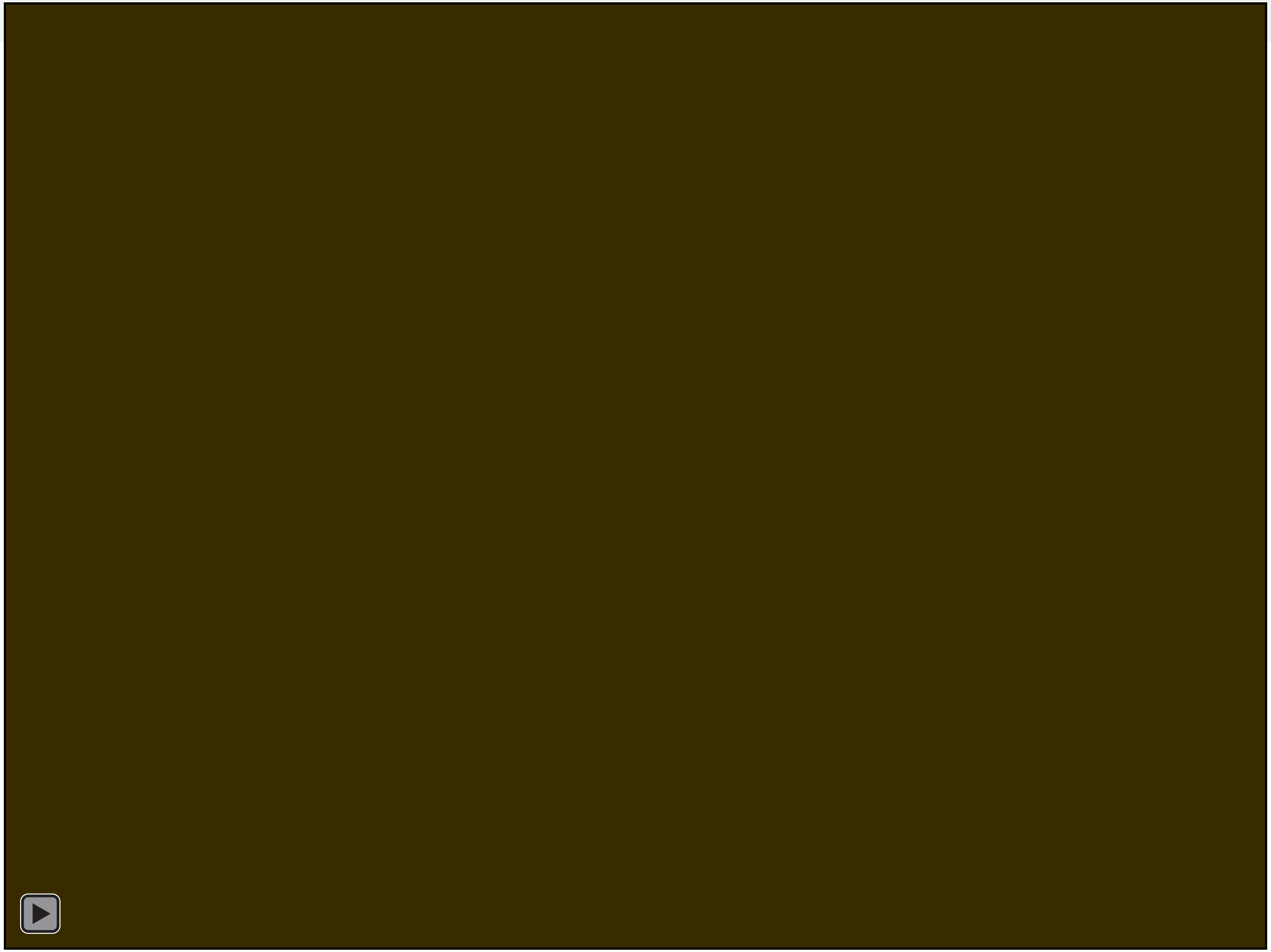
$$\Rightarrow$$

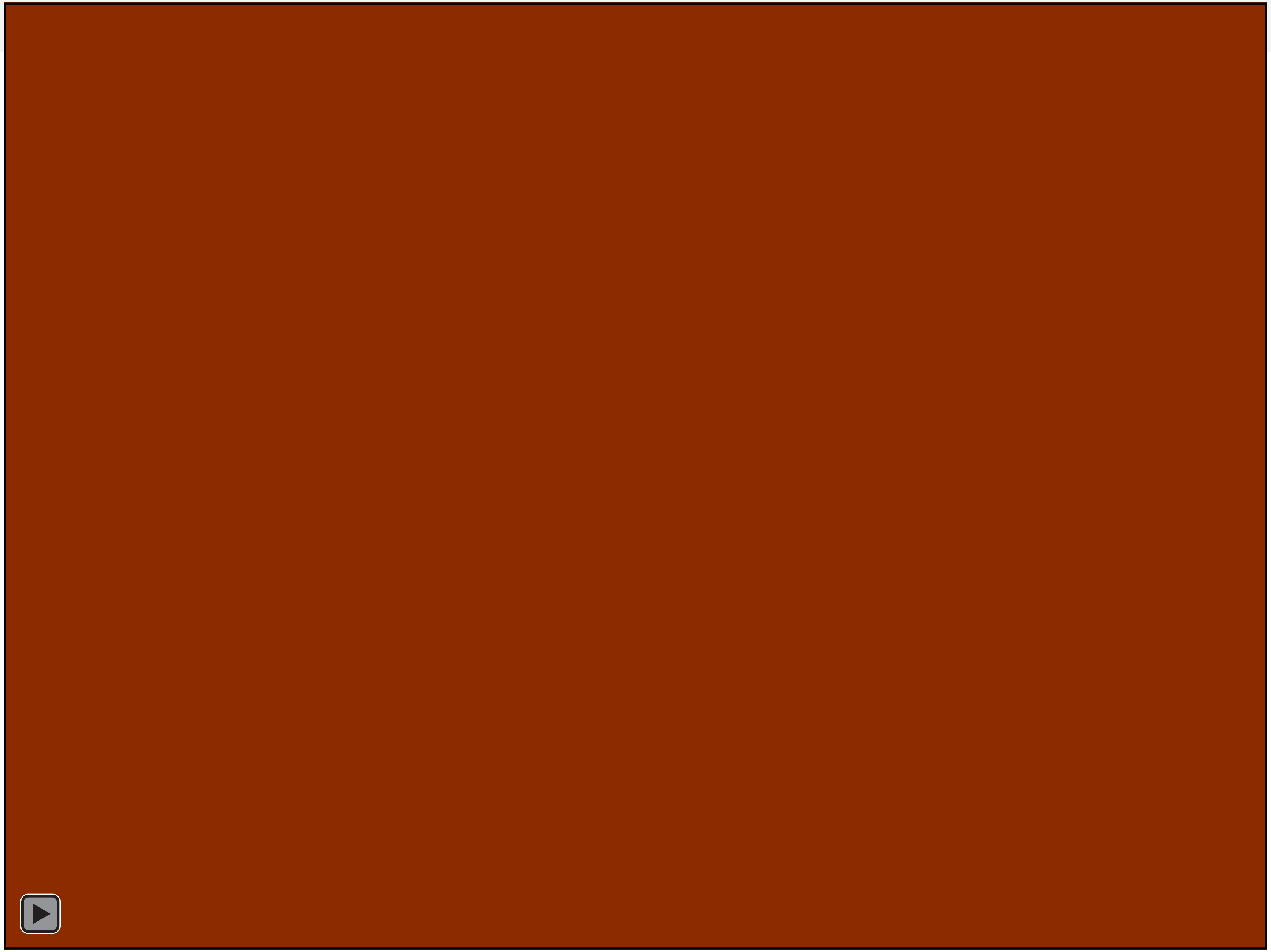
$$\mathbf{F}_G = \mathbf{F}_B \Leftrightarrow m\mathbf{g} = \nabla(\mathbf{m} \cdot \mathbf{B}) \Leftrightarrow$$

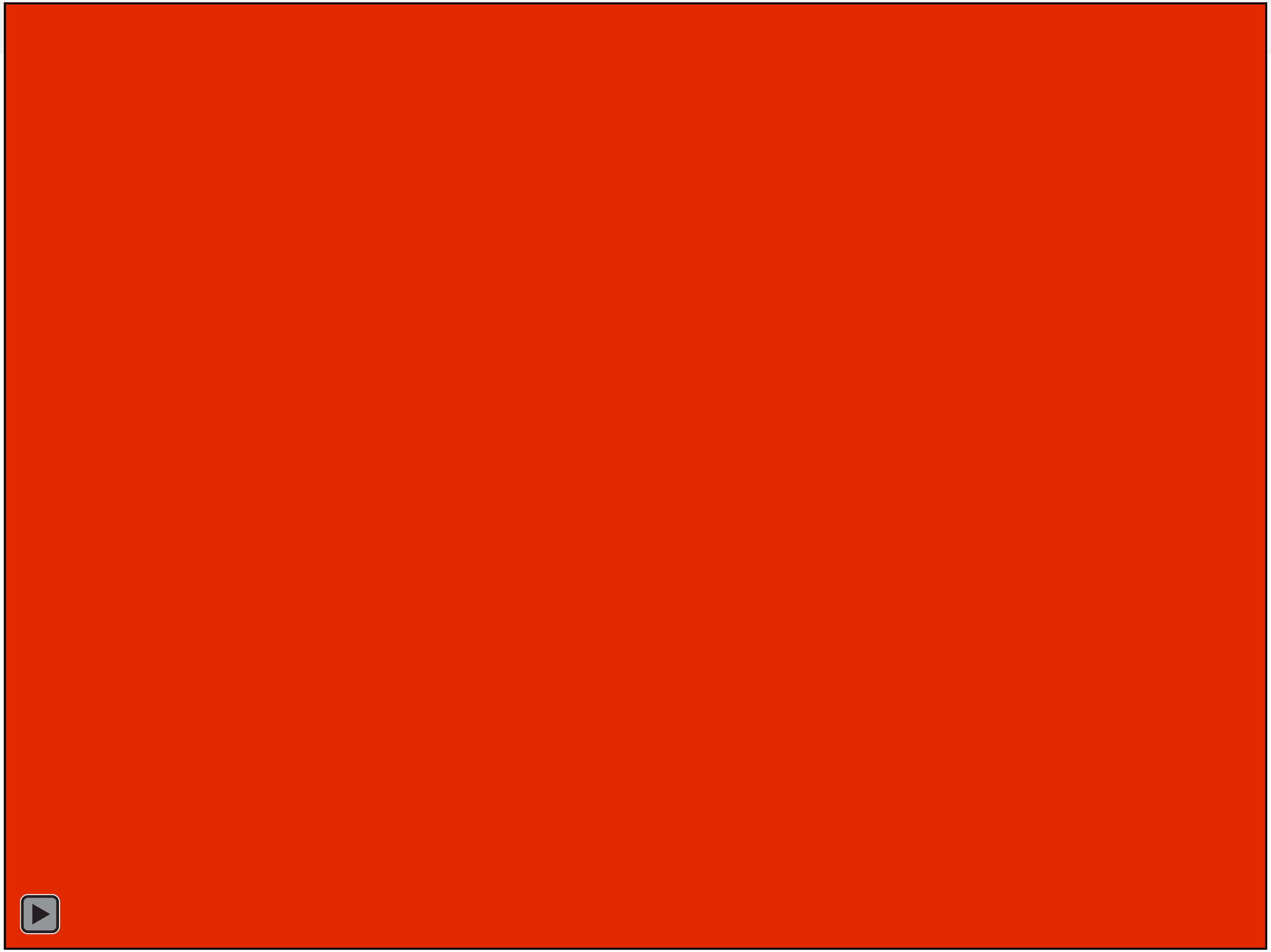
$$\rho V g = \chi_m \frac{B}{\mu_0} V \nabla B \Leftrightarrow \rho g = \chi_m \frac{B}{\mu_0} \nabla B$$

Note: The water droplet, the strawberry, and the frog are diamagnetic objects because they contain water molecules, which are diamagnetic.

http://www.ru.nl/hfml/research/levitation/diamagnetic/

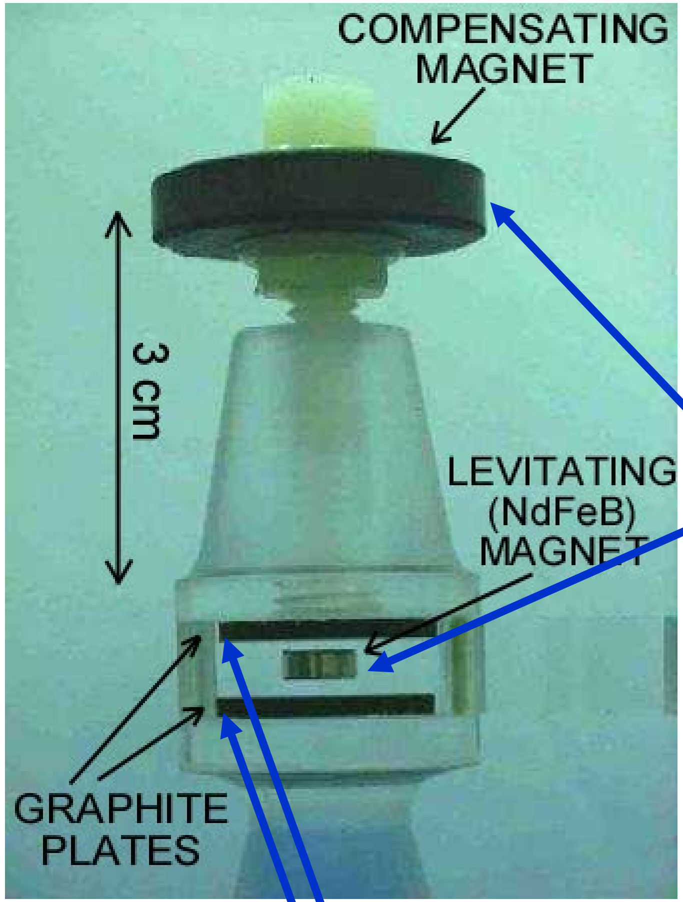






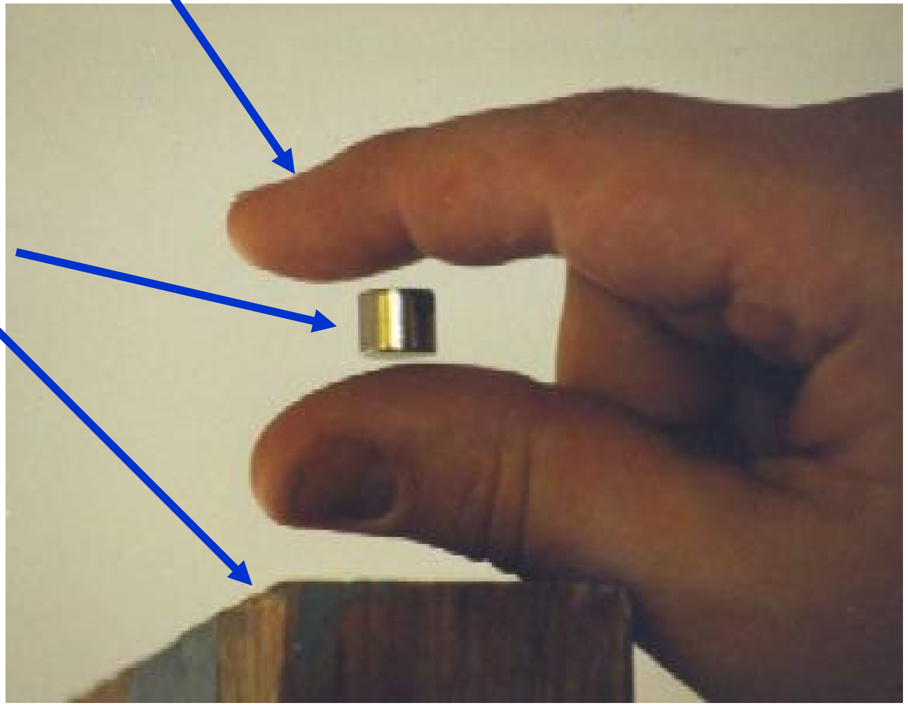
Ferromagnetic object levitation
(stabilized by a diamagnetic material)

Note: The fingers are diamagnetic object because they contain water molecules which are diamagnetic.



Diamagnetic objects (fingers)

Permanent magnets

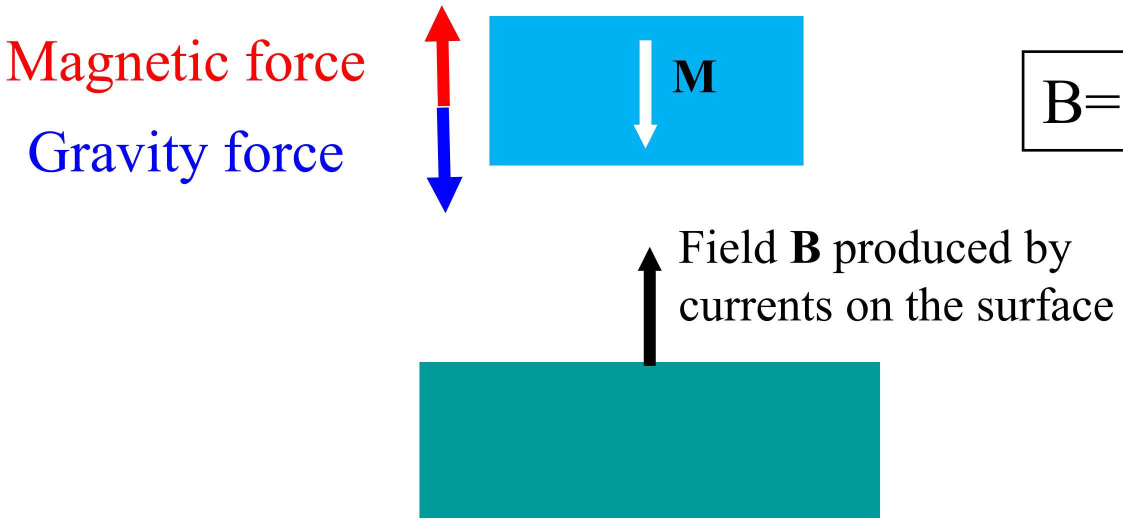


Nature **400**, 323-324 (July 22 1999)

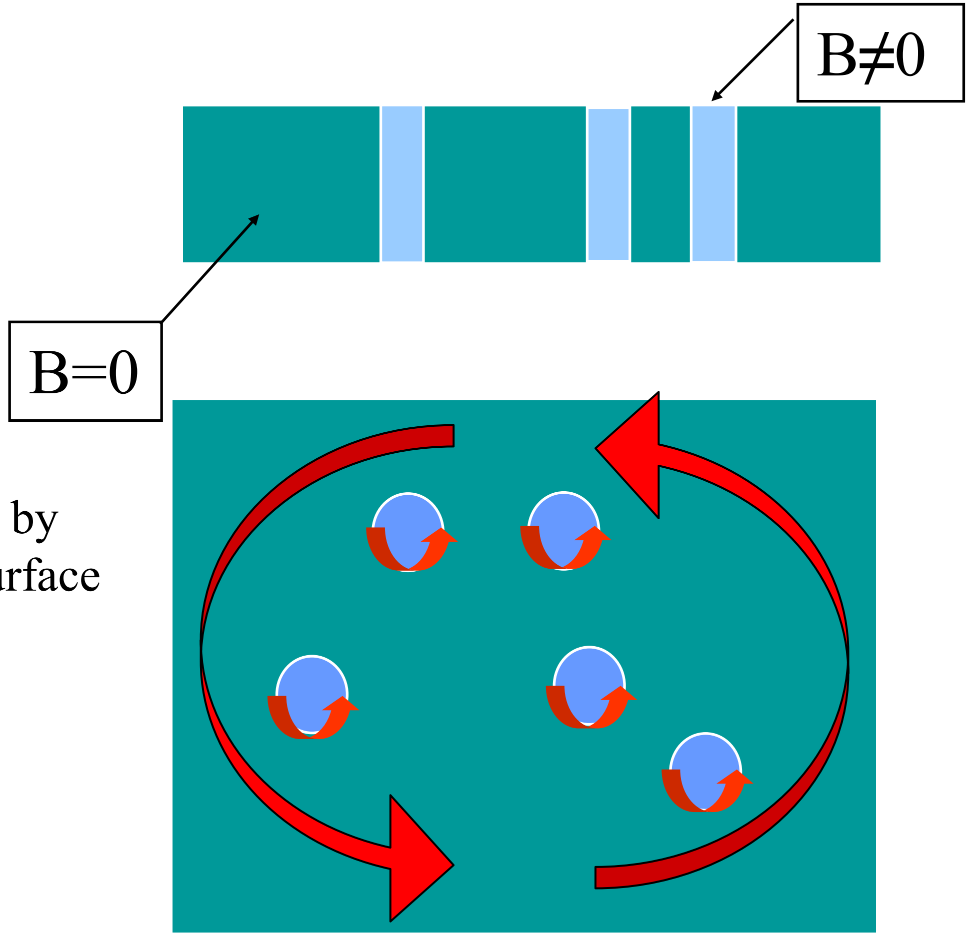
Diamagnetic objects

Ferromagnetic object levitation
(stabilized by a type II superconductor)

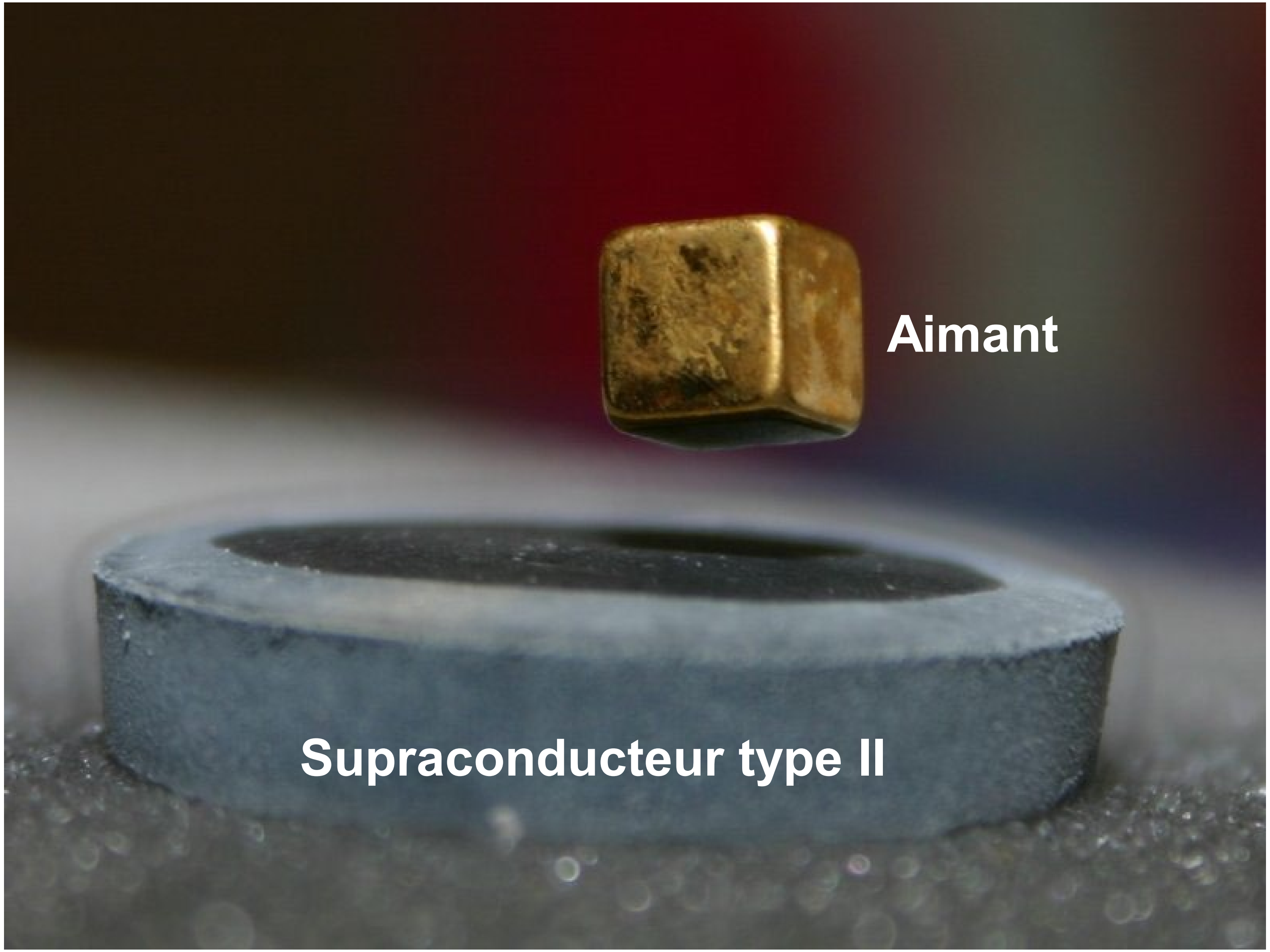
Permanent magnet

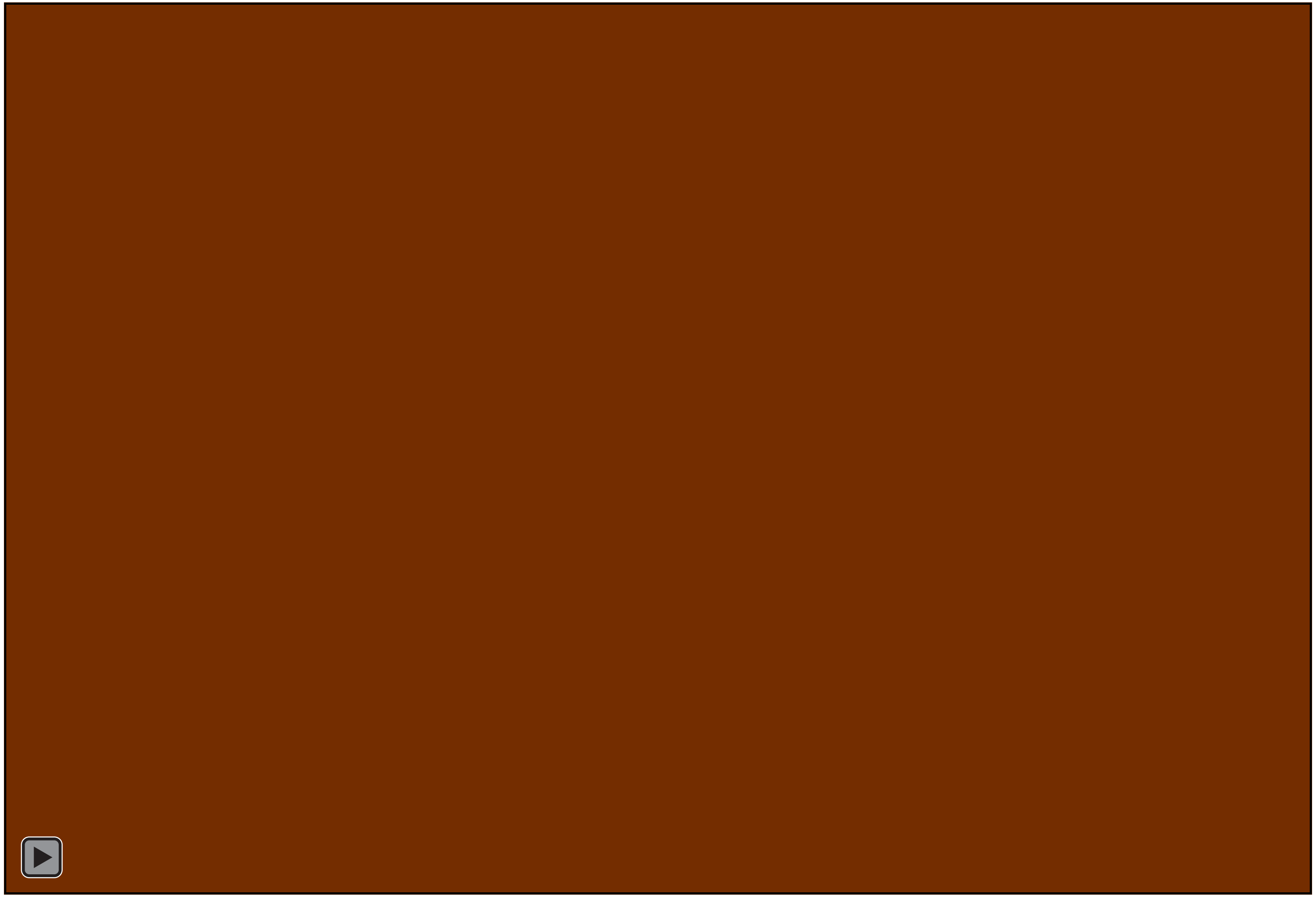


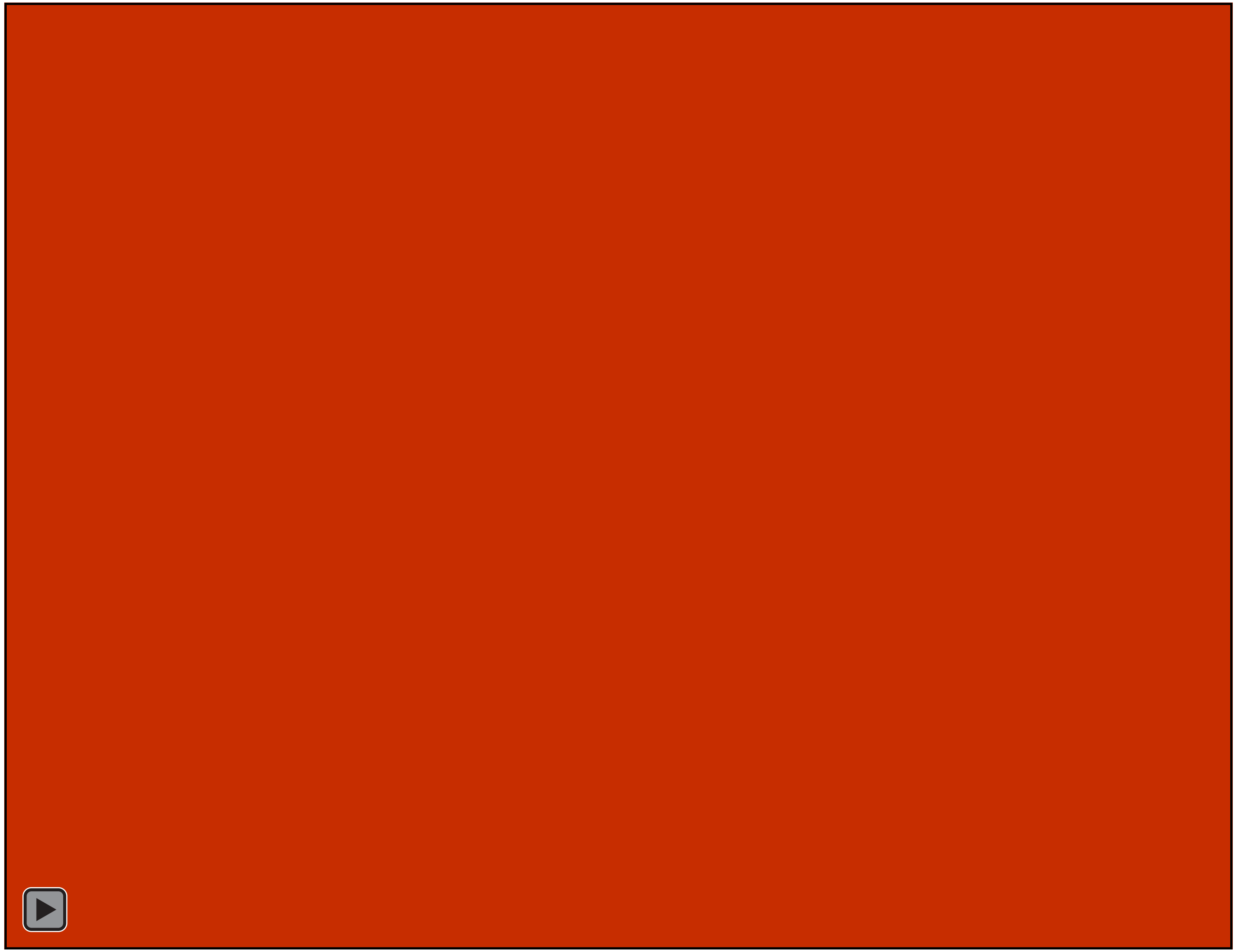
Type II supercodutor
(à 77 K)

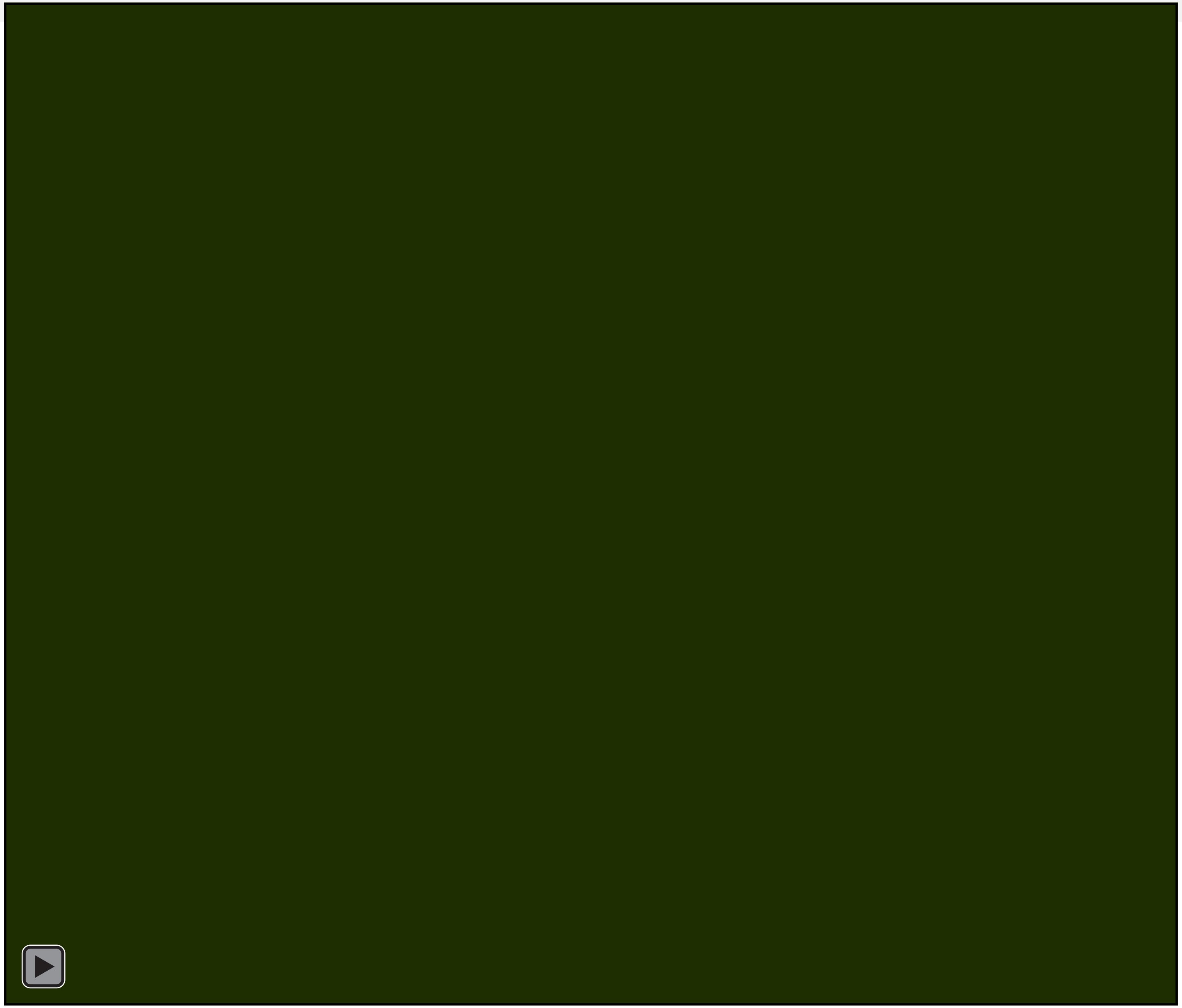


- The field B penetrates only in the vortices
- The surface currents cancel the external field inside the material (except in vortices)


Aimant
Supraconducteur type II







Magnetic levitation (and propulsion)

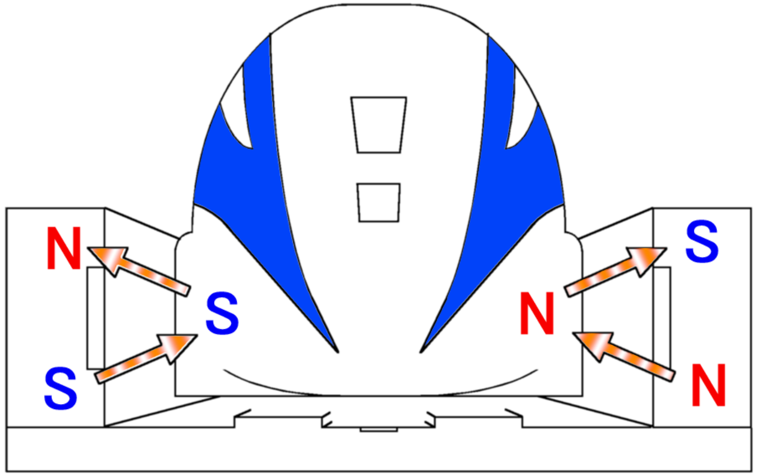
Maglev Train, Shanghai,China

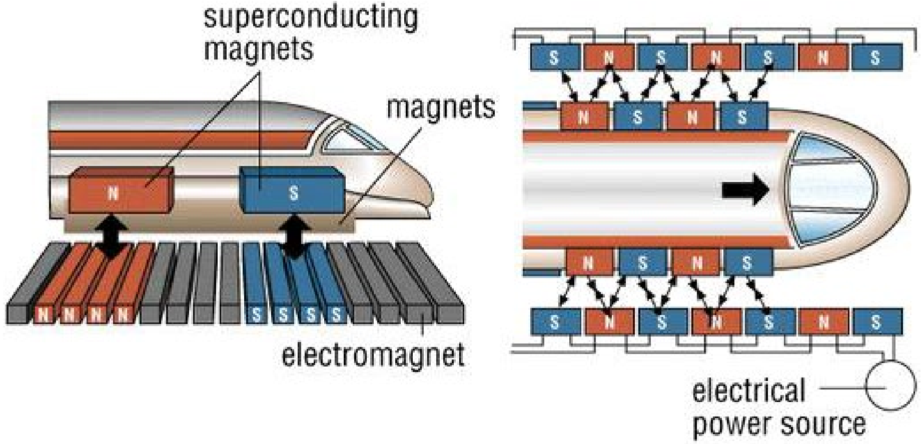
Opening: 2004

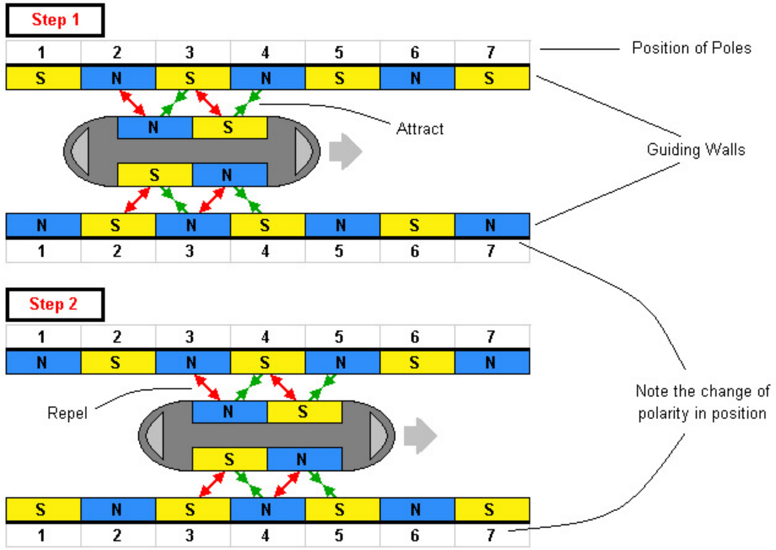
Speed: 430 km/h (operational), 501 km/h (max),

Length:153 m, Width: 3.7 metres, Height: 4.2 m

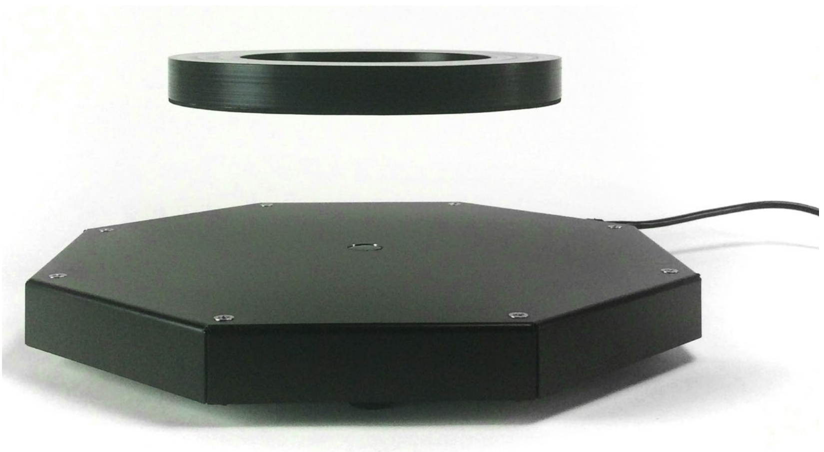
Passengers: 574

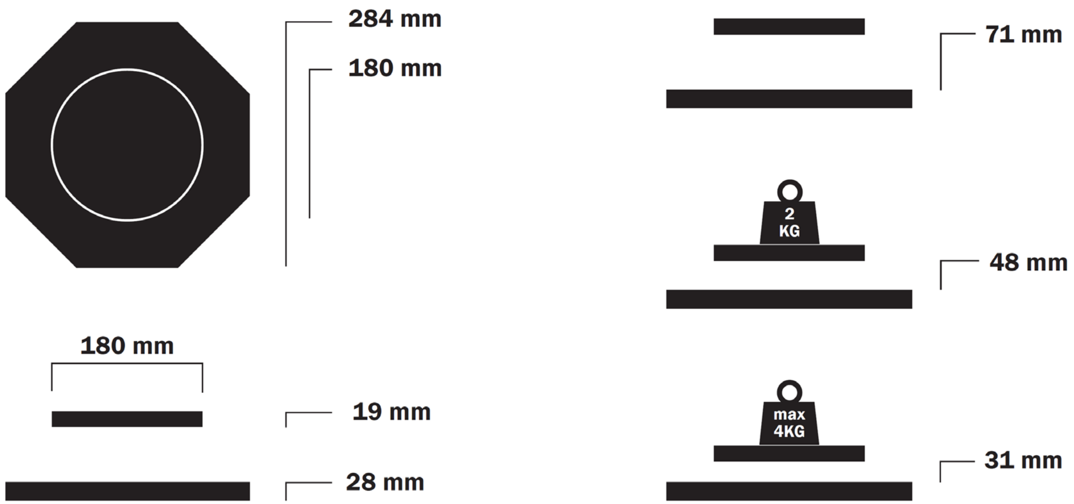






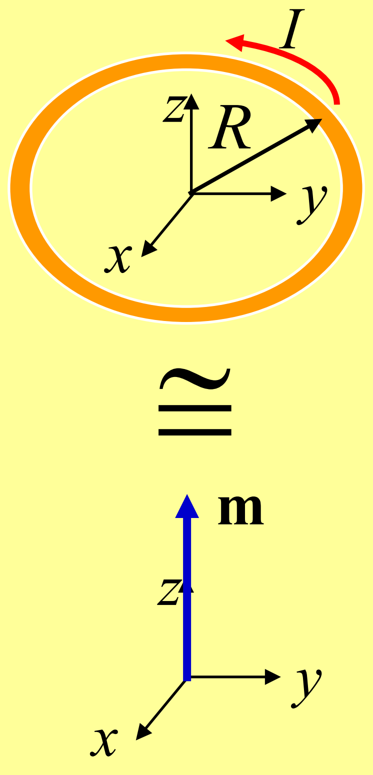
Ferromagnetic object levitation
(stabilized by an array of coils)





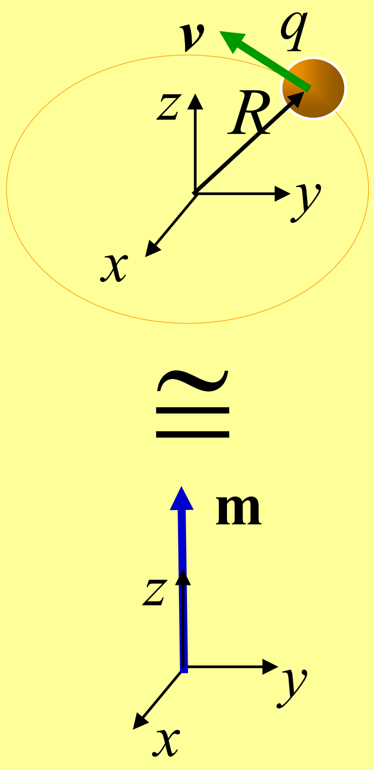
Models of the «magnetic dipole»

Ampère model
(current in a circular coil)



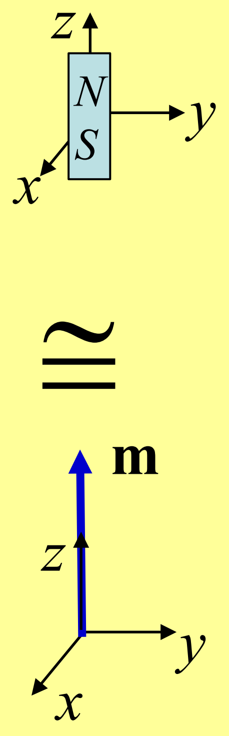
$$\mathbf{m} = I\pi R^2 \hat{\mathbf{z}}$$

(charged particle in a
circular orbit)



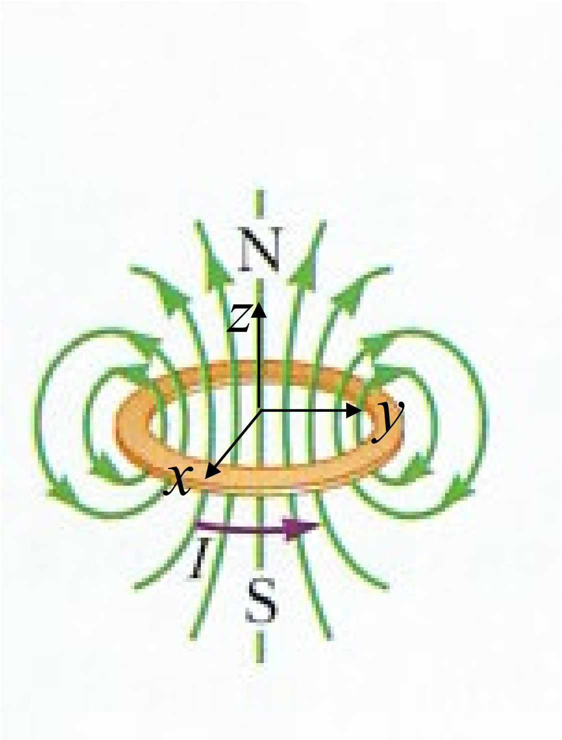
$$\mathbf{m} = q\frac{v}{2}R\hat{\mathbf{z}}$$

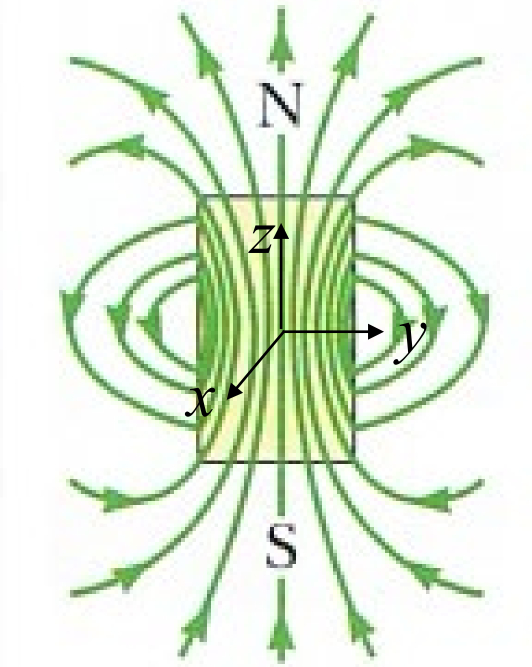
Gilbert model
(permanent magnet)



$$\mathbf{m} = MV\hat{\mathbf{z}}$$

Ampère equivalence principle
beetwen a permanent magnet and a current in circular coil





$\mathbf{m} = I\pi R^2\hat{\mathbf{z}} = IA\hat{\mathbf{z}}$ $\left[\mathrm{Am}^2\right]$

R : coil radius $\left[\mathrm{m}\right]$

I : current $\left[\mathrm{A}\right]$

A : surface of the coil $\left[\mathrm{m}^2\right]$

$\mathbf{m} = MV\hat{\mathbf{z}}$ $\left[\mathrm{Am}^2\right]$

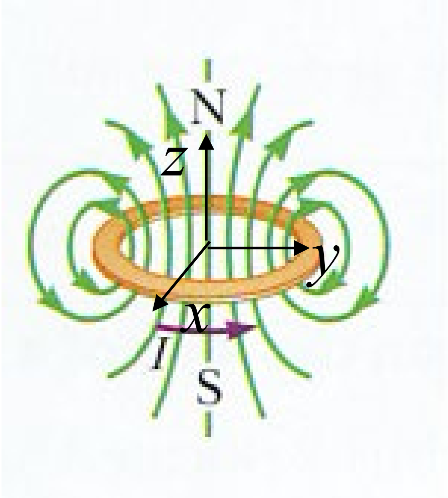
$\mathbf{M}$: Magnetization of the magnet $\left[\mathrm{A/m}\right]$

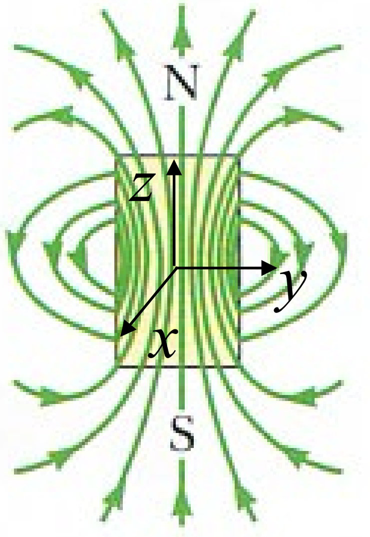
(density of magnetic dipoles)

V : Volume of the magnet $\left[\mathrm{m}^3\right]$

If they have the same magnetic moment **m**:

1. They experience the same force and torque in an external magnetic field (rotation, displacement)
2. They generate the same magnetic field **B** at large distances.





$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi}\oint_C \frac{\hat{\mathbf{t}}\times\hat{\mathbf{r}}}{r^2}dl$$

Analytical solution only for $\mathbf{B}(0,0,z)$ (with the Biot-Savart law):

$$\mathbf{B}(0,0,\mathrm{z}) = \frac{\mu_0 I R^2}{2\left(R^2+z^2\right)^{3/2}}$$

$$\mathbf{m} = I\pi R^2\hat{\mathbf{z}}$$

hence for $|z| >> R$:

$$\mathbf{B}(0,0,z) \cong \frac{\mu_0 I R^2}{2z^3}\hat{\mathbf{z}} = \frac{\mu_0 m}{2\pi z^3}\hat{\mathbf{z}} \propto \frac{m}{z^3}\hat{\mathbf{z}}$$

For $|\mathbf{x}| >>$ dimensions of the magnet:

$$\mathbf{B}(\mathbf{x}) \cong \frac{\mu_0}{4\pi}\frac{3(\mathbf{m}\cdot\hat{\mathbf{x}})\hat{\mathbf{x}}-\mathbf{m}}{|\mathbf{x}|^3}$$

hence for $\mathbf{m}=(0,0,m)$ and $\mathbf{x}=(0,0,z)$:

$$\mathbf{B}(0,0,\mathrm{z}) = \frac{\mu_0}{4\pi}\frac{2m}{z^3}\hat{\mathbf{z}} = \frac{\mu_0}{2\pi}\frac{m}{z^3}\hat{\mathbf{z}} \propto \frac{m}{z^3}\hat{\mathbf{z}}$$

Magnetostatic calculation

Sources

Magnetization (bounded current)	$\mathbf{M}$
Current density (free currents)	$\mathbf{J}$
Virtual currents	$\mathbf{J}_{virt} \equiv \nabla \times \mathbf{M}$
Virtual magnetic charges	$\rho_{mag} \equiv -\nabla \cdot \mathbf{M}$

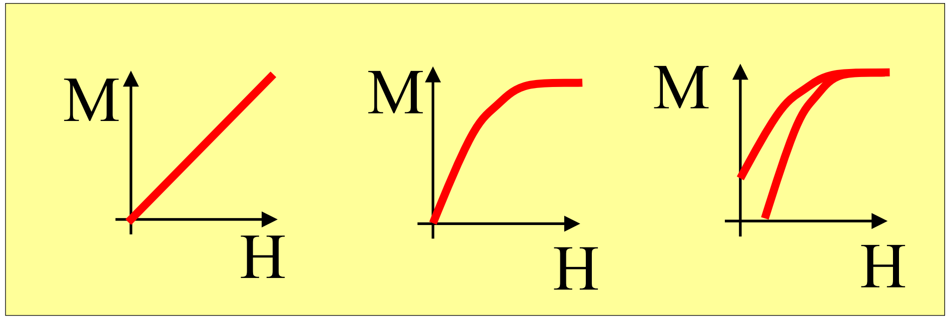
Fields and potentials

Magnetic induction	$\mathbf{B}$
Potential vector	$\mathbf{A}$
Magnetic field	$\mathbf{H}$
Scalar mag. potential	Φ_m

Susceptibility (χ) and permeability (μ)

In general

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} + \ldots.$$



In most of the materials

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \qquad \mathbf{M} = \chi\mathbf{H}$$

$$\mu \equiv \frac{\mathbf{B}}{\mathbf{H}}$$

Consequently,

$$\mathbf{M} = \chi\frac{\mathbf{B}}{(1+\chi)\,\mu_0} \qquad \mathbf{B} = \mu_0(1+\chi)\mathbf{H} = \mu\mathbf{H} \qquad \chi \equiv \frac{\mathbf{M}}{\mathbf{H}}$$

In general:

$$\chi = \chi(\mathbf{H}) \qquad \mu = \mu(\mathbf{H})$$ (eventually anisotropic)

Remanent magnetization = Saturation magnetization

Ideal hard material	Linear material	Non-linear & Hysteresis mat.
$\mathbf{M} = \mathbf{M}_r = \mathbf{M}_s$	$\mathbf{M} = \chi \mathbf{H}$	$\mathbf{M} = \mathbf{M}(\mathbf{H})$
$\rho_{mag} = -\nabla \cdot \mathbf{M}$	$\rho_{mag} = -\dfrac{(\nabla \chi) \cdot \mathbf{H}}{(1+\chi)}$	
$\sigma_{mag} = -\mathbf{n} \cdot (\mathbf{M}_2 - \mathbf{M}_1)$	$\sigma_{mag} = -\dfrac{(\chi_2 - \chi_1)}{(1+(\chi_1+\chi_2)/2)} \dfrac{\mathbf{n} \cdot (\mathbf{H}_1 + \mathbf{H}_2)}{2}$	
$\mathbf{M} = \mathbf{M}_r = \mathbf{M}_s$	$\mathbf{M} = \chi \dfrac{\mathbf{B}}{(1+\chi)\mu_0}$	$\mathbf{M} = \mathbf{M}(\dfrac{\mathbf{B}}{\mu_0} - \mathbf{M})$
$\mathbf{J}_{virt} = \nabla \times \mathbf{M}$	$\mathbf{J}_{virt} = \nabla \chi \times \dfrac{\mathbf{B}}{\mu_0(1+\chi)}$	
$\mathbf{J}_{surf} = \mathbf{n} \times (\mathbf{M}_2 - \mathbf{M}_1)$	$\mathbf{J}_{surf} = \dfrac{(\chi_2 - \chi_1)}{(1+(\chi_1+\chi_2)/2)} \dfrac{\mathbf{n} \times (\mathbf{B}_1 + \mathbf{B}_2)}{2\mu_0}$	

At the interface between material 1 and material 2

Too many quantities and equations ?

Do not miss the
simplicity of the overall scheme !!

Magnetostatic equations

$$\nabla \times \mathbf{H} = \mathbf{J}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

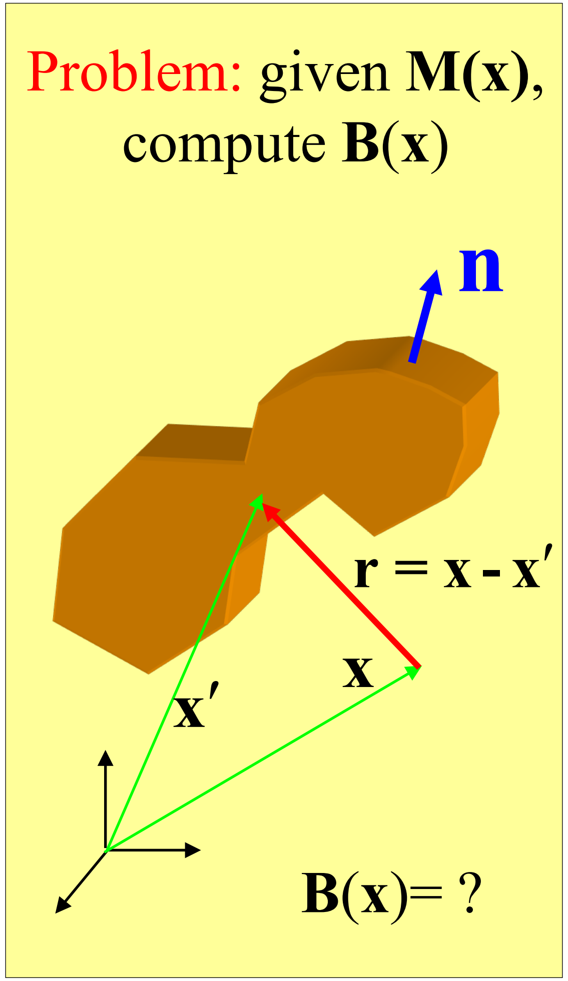
$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{x}') + \nabla \times \mathbf{M}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}'$$

$$\chi \equiv \frac{\mathbf{M}}{\mathbf{H}}$$

$$\mu \equiv \frac{\mathbf{B}}{\mathbf{H}}$$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$$

Hard material ($M=M_s$)



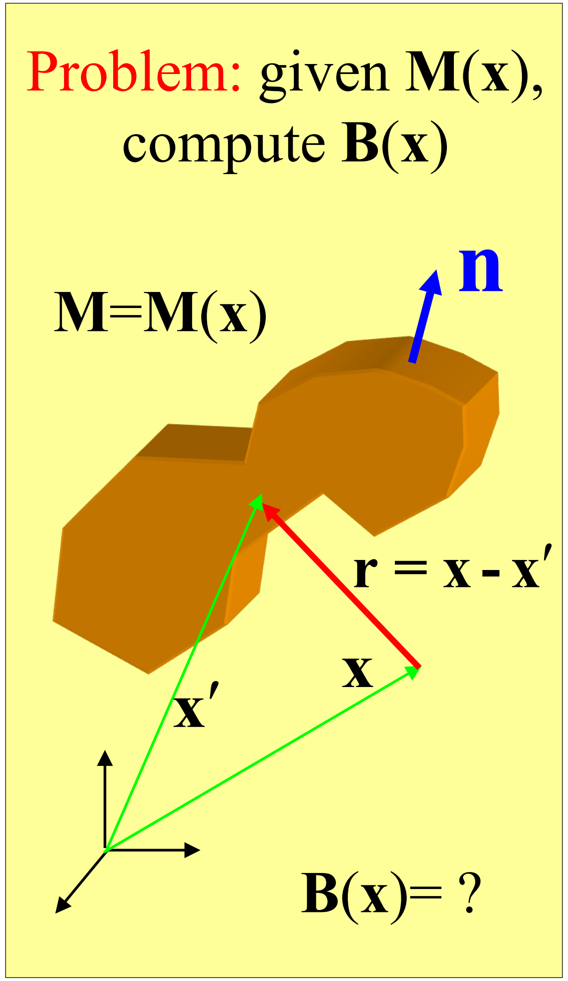
1. Magnetic dipoles method

B of a magnetic dipole $\mathbf{m}_i$

$$\mathbf{B}(\mathbf{x}) = \sum_i \frac{\mu_0}{4\pi} \cdot \frac{3(\mathbf{m}_i \cdot \mathbf{r}_i)\mathbf{r}_i - (\mathbf{r}_i \cdot \mathbf{r}_i) \cdot \mathbf{m}_i}{|\mathbf{r}_i|^5} =$$

$$= \frac{\mu_0}{4\pi} \int_V \frac{3(\mathbf{M}(\mathbf{x}') \cdot \mathbf{r}) \cdot \mathbf{r} - (\mathbf{r} \cdot \mathbf{r}) \cdot \mathbf{M}(\mathbf{x}')}{|\mathbf{r}|^5} d\mathbf{x}'$$

Hard material ($M=M_s$)(2)



2. Virtual currents method

$$\mathbf{J}_{virt} \equiv \nabla \times \mathbf{M}$$

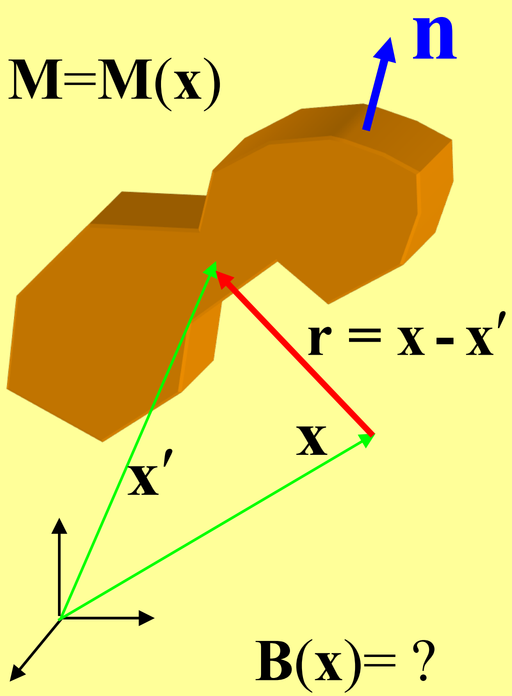
$$\mathbf{J}_{surf} \equiv \mathbf{J}_{virt} \Delta x = -\mathbf{n} \times \mathbf{M}$$

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int_S \frac{\mathbf{J}_{surf}(\mathbf{x}') \times \mathbf{r}}{\mathbf{r}^3} ds$$

Notes: Method 1 is more «physical».
Method 2 requires only integration on a surface

Hard material ($M=M_s$)(3)

Problem: given $\mathbf{M}(\mathbf{x})$, compute $\mathbf{B}(\mathbf{x})$



3. Virtual magnetic charges method

$$\rho_{mag} \equiv -\nabla \cdot \mathbf{M}$$

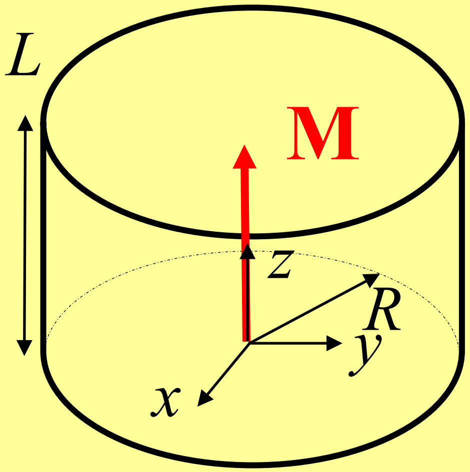
$$\sigma_{mag} \equiv \rho_{mag} \Delta x = \mathbf{n} \cdot \mathbf{M}$$

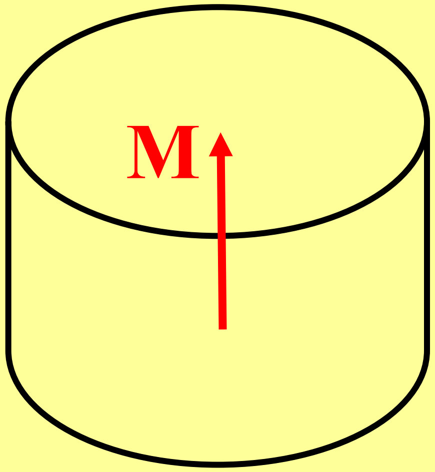
$$\mathbf{H} = \frac{1}{4\pi} \int_S \frac{\sigma_{mag}(\mathbf{x}') \cdot \mathbf{r}}{r^3} ds$$

Hard material ($M=M_s$)(4)

Exemple: cylinder with uniform magnetization

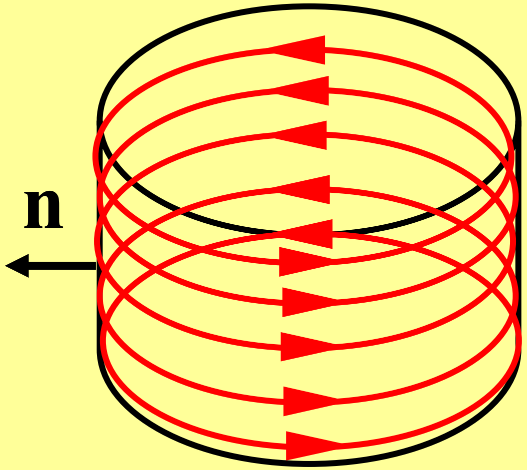
Problem:
$\mathbf{M}(\mathbf{x})$ uniform,
compute $\mathbf{B}(\mathbf{x})$





$$\mathbf{B}(\mathbf{x})=\frac{\mu_0}{4\pi}\int_V \frac{3(\mathbf{M}(\mathbf{x}')\cdot\hat{\mathbf{r}})\cdot\hat{\mathbf{r}}-\mathbf{M}(\mathbf{x}')}{|\mathbf{r}|^3}dV$$

Methode 1



$$\mathbf{J}_{surf}\equiv-\mathbf{n}\times\mathbf{M}$$

$$\mathbf{B}=\frac{\mu_0}{4\pi}\int_S \frac{\mathbf{J}_{surf}(\mathbf{x}')\times\mathbf{r}}{|\mathbf{r}|^3}ds$$

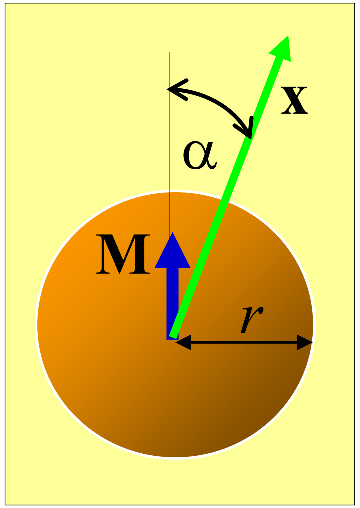
Methode 2

Pour $z >> R, L$ et $\mathbf{r}\cong(0,0,z)$

$$\mathbf{B}(0,0,z)=\frac{\mu_0}{4\pi}\int_V \frac{3(\mathbf{M}(\mathbf{x}')\cdot\hat{\mathbf{r}})\cdot\hat{\mathbf{r}}-\mathbf{M}(\mathbf{x}')}{|\mathbf{r}|^3}dV\cong\frac{\mu_0}{4\pi}\int_V\frac{2Mz^2\hat{\mathbf{z}}}{|z|^5}dV\cong\frac{\mu_0}{2\pi}\frac{1}{z^3}M\pi R^2L\hat{\mathbf{z}}=\frac{\mu_0}{2}\frac{MR^2L}{z^3}\hat{\mathbf{z}}=\frac{\mu_0}{2\pi}\frac{\mathbf{m}}{z^3}$$

$$\mathbf{B}(0,0,z)=\frac{\mu_0}{4\pi}\int_S\frac{\mathbf{J}_{surf}(\mathbf{x}')\times\hat{\mathbf{r}}}{|\mathbf{r}|^2}ds\cong\frac{\mu_0 I_b R^2}{2z^3}=\frac{\mu_0}{2\pi}\frac{\mathbf{m}}{z^3}$$

Hard material ($M=M_s$)(5)



1. Sphere with the scalar magnetic potential approach

$$\Phi_M(\mathbf{x}) = \frac{1}{4\pi} \cdot \int_V \frac{-\nabla \cdot \mathbf{M}(\mathbf{x}')}{|\mathbf{x}|} \cdot d\mathbf{x}$$

Into the sphere:

$$\mathbf{H} = -\frac{1}{3}\mathbf{M}$$

$$\mathbf{B} = \mu_0 \cdot (\mathbf{H} + \mathbf{M}) = +\frac{2}{3}\mathbf{M}$$

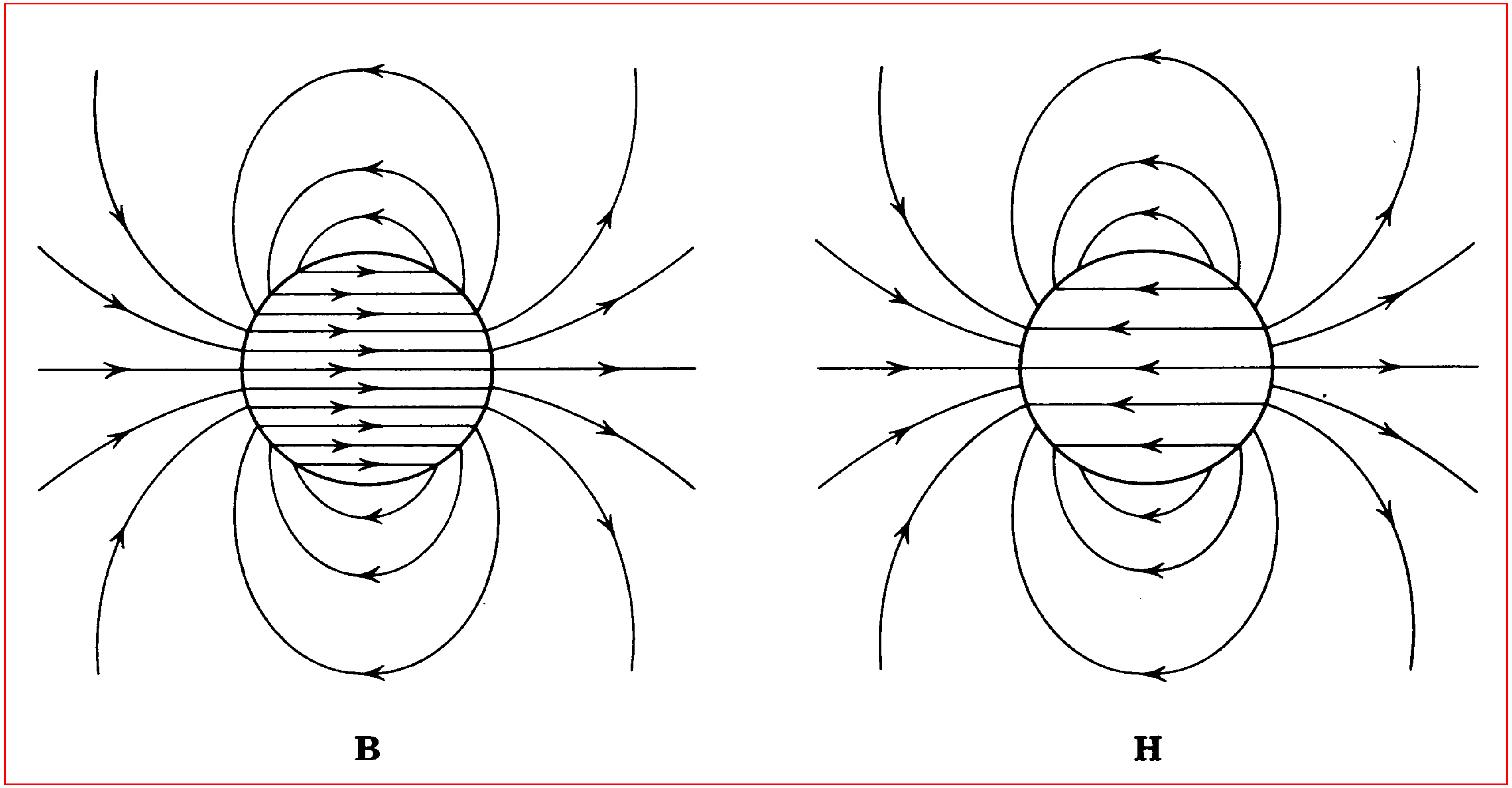
As for magnetic dipole **m** in **x**=0

Out of the sphere:

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \frac{1}{4\pi} \frac{3(\mathbf{m} \cdot \mathbf{x})\mathbf{x} - (\mathbf{x} \cdot \mathbf{x}) \cdot \mathbf{m}}{|\mathbf{x}|^5}$$

$$\mathbf{m} = (\frac{4\pi r^3}{3})\mathbf{M}$$

Hard material ($M=M_s$)(6)

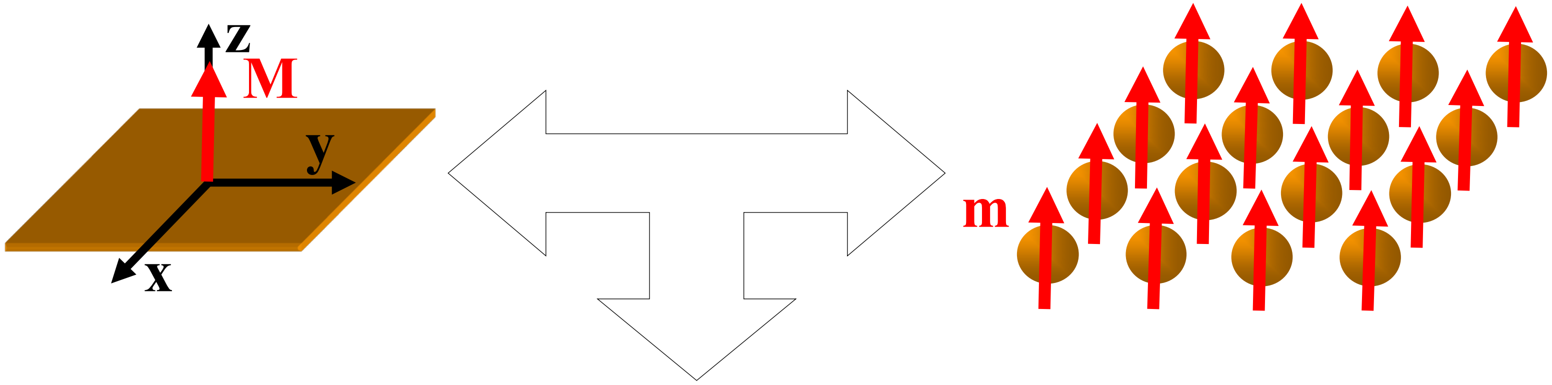


Into the sphere: $\mathbf{B}, \mathbf{H}$ *uniform*

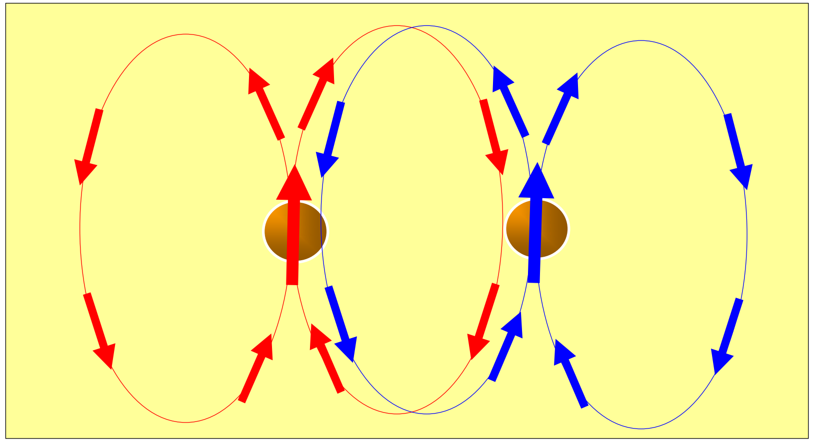
At the surface: B_n, H_t *continuous*

Hard material ($M=M_s$)(7)

Thin plate perp.:

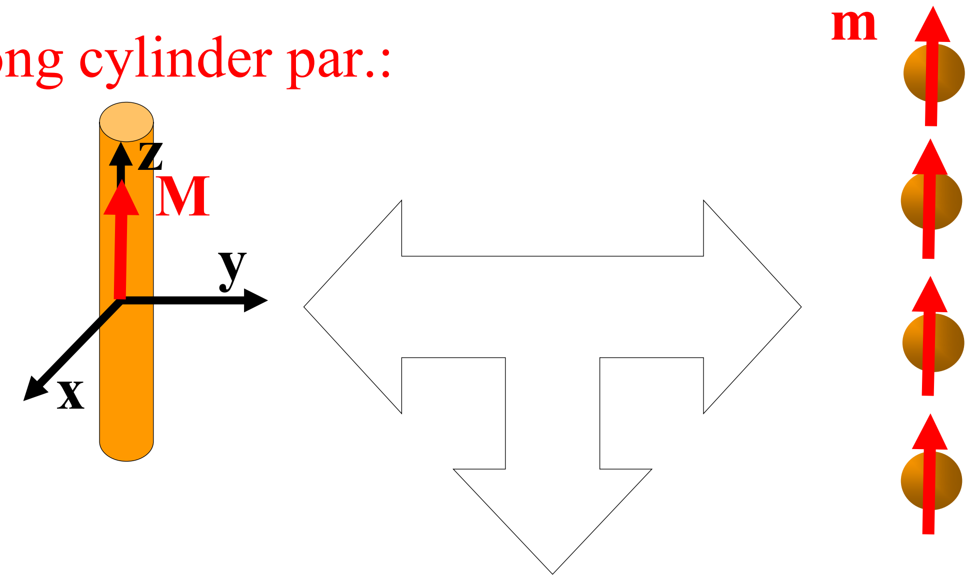


$$\mathbf{B}(\mathbf{x}) = \sum_i \frac{\mu_0}{4\pi} \cdot \frac{3(\mathbf{m}_i \cdot \mathbf{r}_i)\mathbf{r}_i - (\mathbf{r}_i \cdot \mathbf{r}_i) \cdot \mathbf{m}_i}{\left|\mathbf{r}_i\right|^5} \cong 0$$

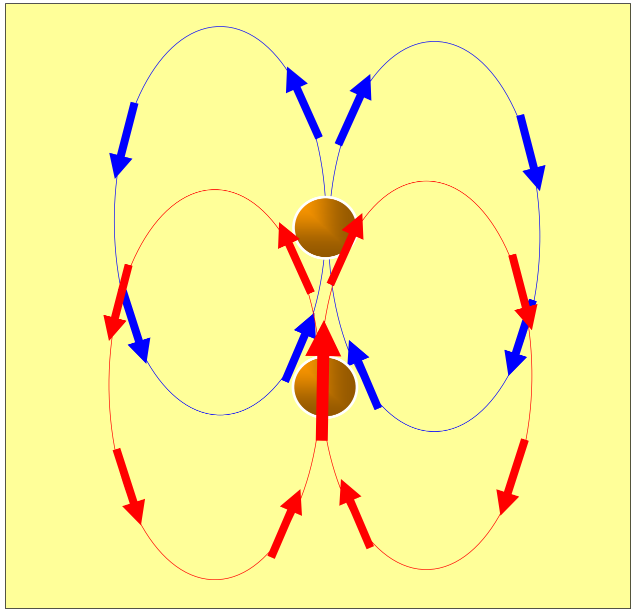


Hard material ($M=M_s$)(8)

Long cylinder par.:



$$\mathbf{B}(\mathbf{x}) = \sum_i \frac{\mu_0}{4\pi} \cdot \frac{3(\mathbf{m}_i \cdot \mathbf{r}_i)\mathbf{r}_i - (\mathbf{r}_i \cdot \mathbf{r}_i) \cdot \mathbf{m}_i}{|\mathbf{r}_i|^5} \neq 0$$



Linear material (M=χH)

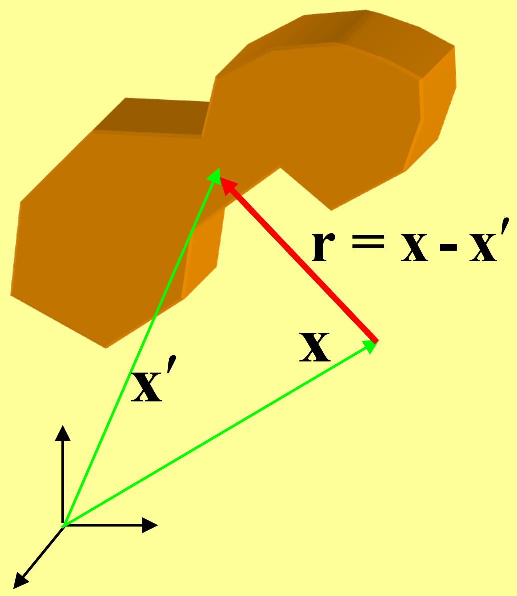
1. Solution with virtual magnetic charges

$$\mathbf{H}(\mathbf{x}) = \mathbf{H}_a + \frac{1}{4\pi}\int_S \frac{\sigma_{mag}(\mathbf{x}')\cdot\mathbf{r}}{r^3}ds$$

2. Solution with virtual currents

$$\mathbf{B}(\mathbf{x}) = \mathbf{B}_a(\mathbf{x}) + \frac{\mu_0}{4\pi}\int_S \frac{\mathbf{J}_{surf}(\mathbf{x}')\times\mathbf{r}}{r^3}ds$$

Note: With the virtual charges and currents we have to perform integrals on the object surface instead of in the object volume.

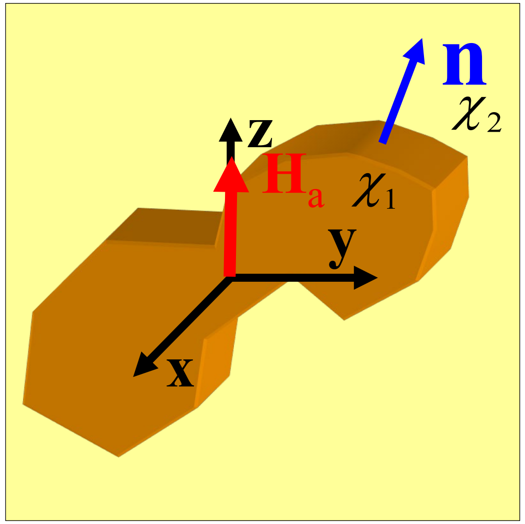


χ uniform in the object

Problem:
Given $\mathbf{B}_a$ and χ,
Compute $\mathbf{B}$(x)

Linear material (M=χH) (2)

1. Virtual magnetic charges method



$$\nabla \cdot \mathbf{B} = 0 \;\Rightarrow\; \mu_0 \nabla \cdot \mathbf{H} + \mu_0 \nabla \cdot (\chi \mathbf{H}) = 0$$

$$\Rightarrow \rho_{mag} \equiv -\nabla \cdot \mathbf{M} = \nabla \cdot \mathbf{H} = -\frac{(\nabla \chi) \cdot \mathbf{H}}{(1+\chi)}$$

$$\sigma_{mag} \equiv \rho_{mag} \Delta x = -\frac{(\chi_2 - \chi_1)}{(1+(\chi_1+\chi_2)/2)} \frac{\mathbf{n} \cdot (\mathbf{H}_1 + \mathbf{H}_2)}{2}$$

Start values: $\sigma_{mag\,0} = 0$ $\quad \mathbf{H}_0(\mathbf{x}) = \mathbf{H}_a$ $\quad n = 0$

Iteration:

$$\sigma_{mag\,n+1} = -\frac{(\chi_2 - \chi_1)}{(1+(\chi_1+\chi_2)/2)} \cdot \left(\frac{\mathbf{n} \cdot (\mathbf{H}_1 + \mathbf{H}_2)}{2} \right)_n$$

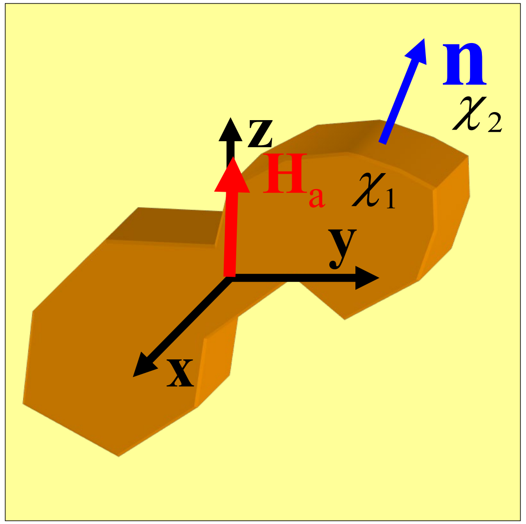
$$\mathbf{H}_{n+1} = \mathbf{H}_a + \frac{1}{4\pi} \int_S \frac{\sigma_{mag\,n+1}(\mathbf{x}') \cdot \mathbf{r}}{r^3} ds$$

$n = n+1$

Conv.

Linear material (M=χH) (3)

2. Virtual currents method



$$\mathbf{J}_{virt} \equiv \nabla\times\mathbf{M} = \nabla\times(\chi\mathbf{H}) = \chi\underbrace{(\nabla\times\mathbf{H})}_{=0} + (\nabla\chi)\times\mathbf{H} =$$

$$= (\nabla\chi)\times\mathbf{H} = (\nabla\chi)\times\frac{\mathbf{B}}{\mu_0(1+\chi)}$$

$$\mathbf{J}_{surf} \equiv \mathbf{J}_{virt}\Delta x = \frac{(\chi_2-\chi_1)}{(1+(\chi_1+\chi_2)/2)}\left(\mathbf{n}\times\frac{\mathbf{B}_1+\mathbf{B}_2}{2\mu_0}\right)$$

Start values: $\mathbf{J}_{surf\,0} = 0$ $\qquad \mathbf{B}_0(\mathbf{x}) = \mathbf{B}_a \qquad n=0$

Iteration: $$\mathbf{J}_{surf_{n+1}} = \frac{(\chi_2-\chi_1)}{(1+(\chi_1+\chi_2)/2)}\left(\mathbf{n}\times\frac{\mathbf{B}_1+\mathbf{B}_2}{2\mu_0}\right)_n$$

$$\mathbf{B}_{n+1} = \mathbf{B}_a + \frac{\mu_0}{4\pi}\int_S \frac{\mathbf{J}_{surf\,n+1}(\mathbf{x}')\times\mathbf{r}}{r^3}dS$$

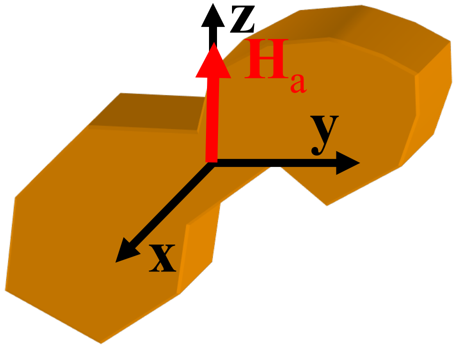
$n=n+1$

Conv.

Linear material ($M=\chi H$) (4)

Problem:

Given a **linear material** placed in a uniform externally applied $\mathbf{H}_a$, which is **H**, **M** and **B** into the object ?



Answer:

If the shape of the object is arbitrary: complicated numerical problem.

If the shape of the object is ellipsoidal: simple solutions available in literature

Linear material (M=χH) (5)

In ellipsoids: **M** and **H** are uniform under a uniform applied $\mathbf{H}_a$, and:

$$\mathbf{H} = \mathbf{H}_a - \tilde{N}\mathbf{M}$$

$$\tilde{N} = \begin{pmatrix} N_x & 0 & 0 \\ 0 & N_y & 0 \\ 0 & 0 & N_z \end{pmatrix} \qquad N_x + N_y + N_z = 1$$

$\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ along the principal ellipsoid axis

$\tilde{N}$: Demagnetizing tensor

If $\mathbf{H}_a$ is along one of the principal axis i:

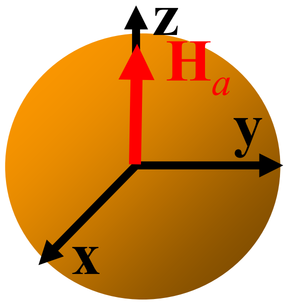
$$\boxed{\begin{array}{ll} H_j = H_{aj} - N_j M_j & j = i \\ H_j = 0 & j \neq i \end{array}}$$

N_x, N_y, N_z for ellipsoids are available in literature (J.A. Osborn,, Phys. Rev. 67, 351, 1945)

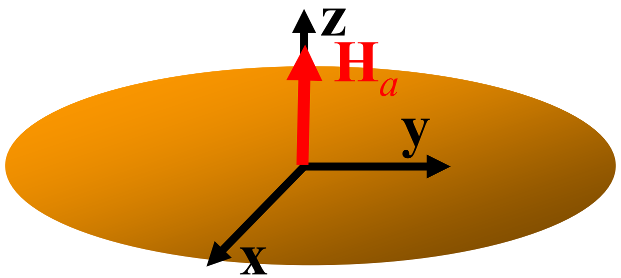
Linear material ($M=\chi H$) (6)

Ellipsoids

Sphere

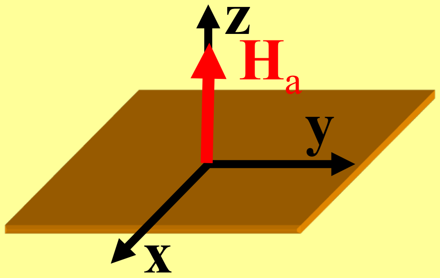


General ellipsoid

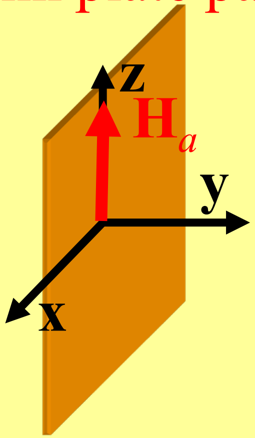


Pseudo-ellipsoids

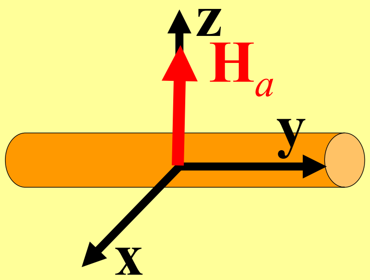
Thin plate perp.:



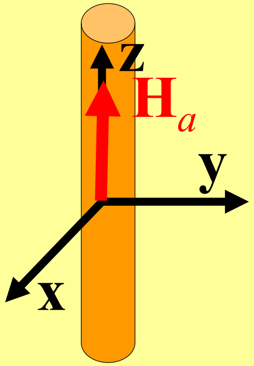
Thin plate par.:



Long cylinder perp.:

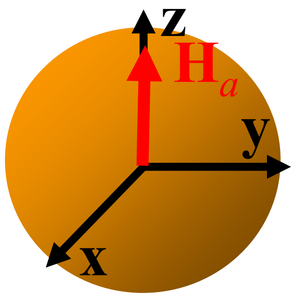


Long cylinder par.:



Linear material (M=χH) (7)

Sphere:



$$\boxed{\mathbf{H} = \mathbf{H}_a - \tilde{N}\mathbf{M}}$$

$$N_x = 1/3 \qquad H_x = 0$$

$$N_y = 1/3 \qquad H_y = 0$$

$$N_z = 1/3 \qquad H_z = H_a - (1/3)M_z$$

$$\mathbf{M} = \chi\mathbf{H}$$

$$\mathbf{B} = \mu_0\left(\mathbf{H} + \mathbf{M}\right)$$

$$M_x = 0 \qquad H_x = 0 \qquad B_x = 0$$

$$M_y = 0 \qquad H_y = 0 \qquad B_y = 0$$

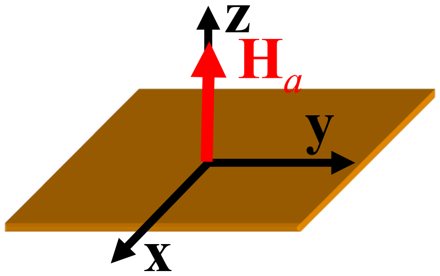
$$M_z = \frac{3\chi}{3+\chi}H_a \qquad H_z = \frac{3}{3+\chi}H_a \qquad B_z = \frac{3(1+\chi)}{3+\chi}\mu_0 H_a$$

Note: For $\chi \rightarrow \infty$, B and M saturates whereas H→0

EPFL

Linear material (M=χH) (8)

Thin plate perp.:



$$N_x = 0 \qquad N_y = 0 \qquad N_z = 1$$

$$H_x = 0 \qquad H_y = 0 \qquad H_z = H_a - M_z$$

$$\mathbf{M} = \chi\mathbf{H}$$

$$\mathbf{B} = \mu_0\left(\mathbf{H} + \mathbf{M}\right)$$

$$M_x = 0 \qquad M_y = 0 \qquad M_z = \frac{\chi}{1+\chi}H_a$$

$$H_x = 0 \qquad H_y = 0 \qquad H_z = \frac{1}{1+\chi}H_a$$

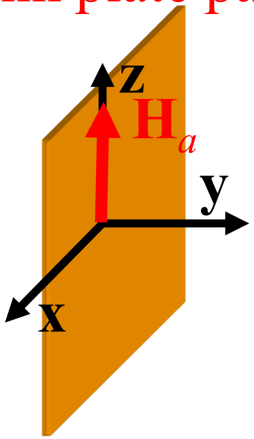
$$B_x = 0 \qquad B_y = 0 \qquad B_z = \mu_0 H_a$$

Note: Since $B_{n1}=B_{n2}$, **B** over the plate is also equal to $\mu_0\mathbf{H}_a$, even for $\chi\to\infty$

Note: For $\chi\to\infty$, M saturates whereas H$\to$0

Linear material (M=χH) (9)

Thin plate par.:



$$N_x = 0 \qquad H_x = 0$$
$$N_y = 1 \qquad H_y = 0$$
$$N_z = 0 \qquad H_z = H_a$$

$$\mathbf{M} = \chi \mathbf{H}$$

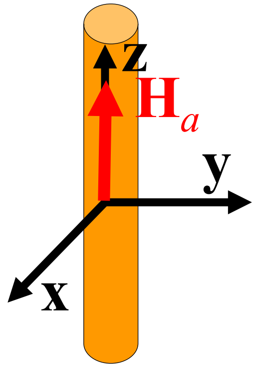
$$\mathbf{B} = \mu_0 \left(\mathbf{H} + \mathbf{M} \right)$$

$$M_x = 0 \qquad H_x = 0 \qquad B_x = 0$$
$$M_y = 0 \qquad H_y = 0 \qquad B_y = 0$$
$$M_z = \chi H_a \qquad H_z = H_a \qquad B_z = \mu_0 (1+\chi) H_a$$

Note: Since B_{n1}=B_{n2}, **B** at the bottom and top plate edges can be much larger than $\mu_0 \mathbf{H}_a$ (if χ>>1).

Linear material (M=χH) (10)

Long cylinder par.:



$$N_x = 1/2 \qquad H_x = 0$$
$$N_y = 1/2 \qquad H_y = 0$$
$$N_z = 0 \qquad H_z = H_a$$

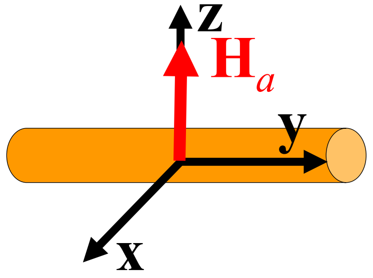
$$\mathbf{M} = \chi \mathbf{H}$$
$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M})$$

$$M_x = 0 \qquad H_x = 0 \qquad B_x = 0$$
$$M_y = 0 \qquad H_y = 0 \qquad B_y = 0$$
$$M_z = \chi H_a \qquad H_z = H_a \qquad B_z = \mu_0 (1+\chi) H_a$$

Note: Since $B_{n1}=B_{n2}$, **B** at the bottom and top plate edges can be much larger than $\mu_0 \mathbf{H}_a$ (if $\chi >> 1$).

Linear material (M=χH) (11)

Long cylinder perp.:



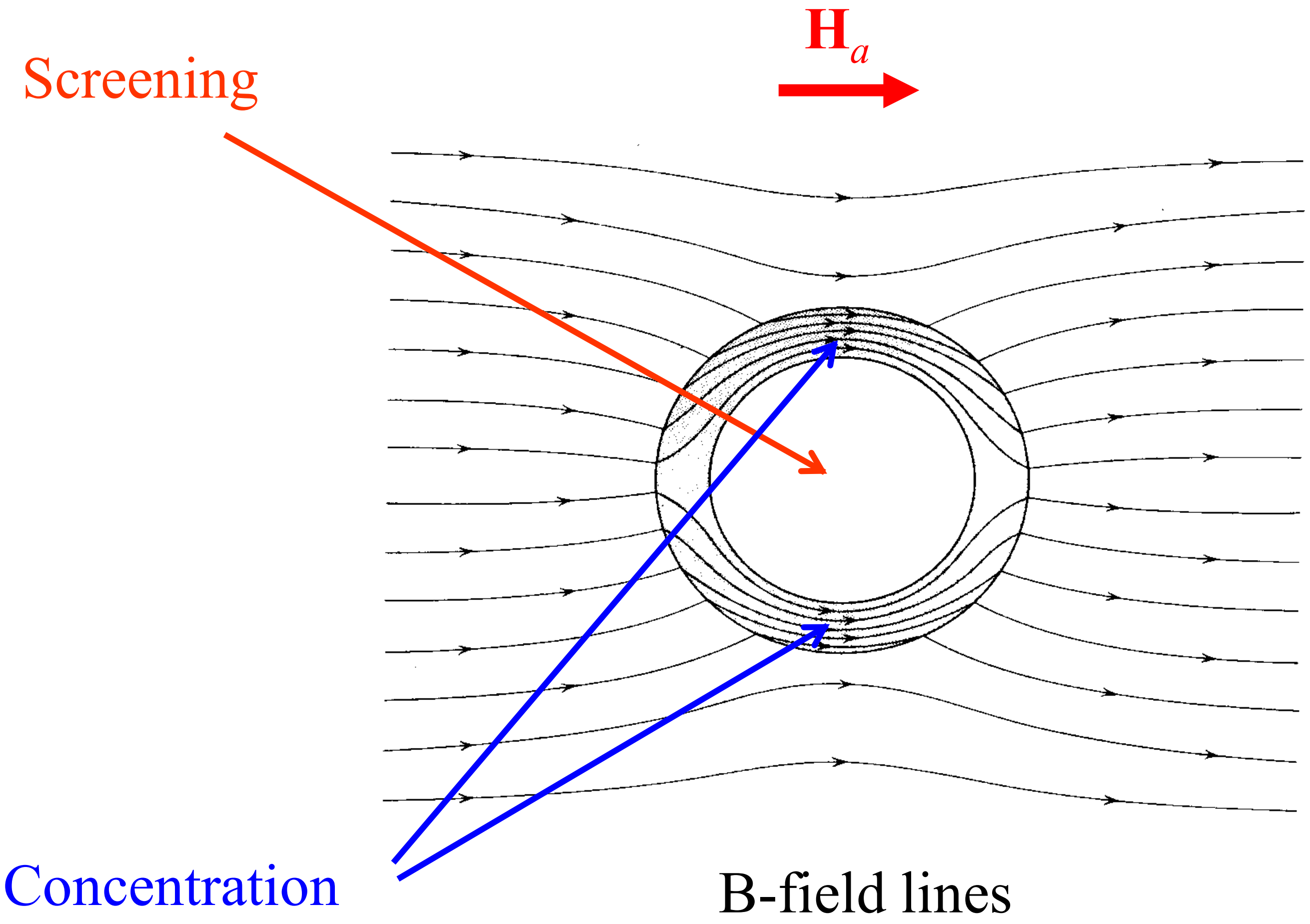
$$N_x = 1/2 \qquad H_x = 0$$
$$N_y = 0 \qquad H_y = 0$$
$$N_z = 1/2 \qquad H_z = H_a - (1/2)M_z$$

$$\mathbf{M} = \chi\mathbf{H}$$
$$\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M})$$

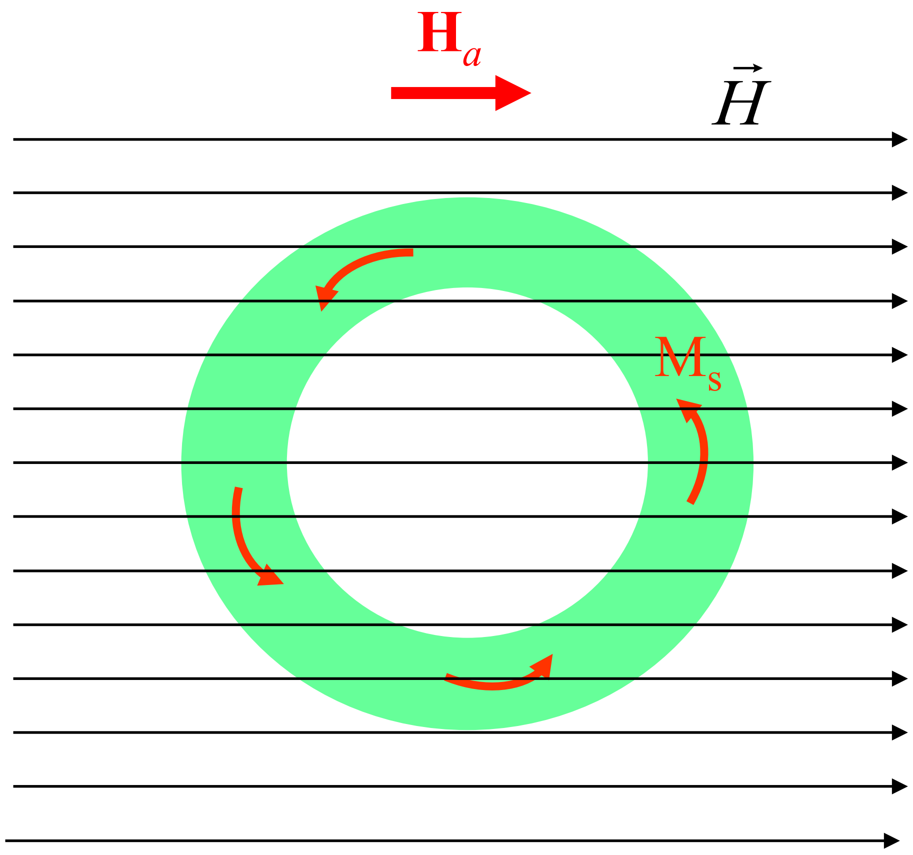
$$M_x = 0 \qquad H_x = 0 \qquad B_x = 0$$
$$M_y = 0 \qquad H_y = 0 \qquad B_y = 0$$
$$M_z = \frac{2\chi}{2+\chi}H_a \qquad H_z = \frac{2}{2+\chi}H_a \qquad B_z = \mu_0\frac{2\chi+2}{2+\chi}H_a$$

Note: For χ→∞, B and M saturates whereas H→0

Ring of linear material ($\mathbf{M}=\chi\mathbf{H}$): screening



Ring of a magnetized hard material: no effect



No virtual magnetic charges

$$\mathbf{H} = \mathbf{H}_a$$

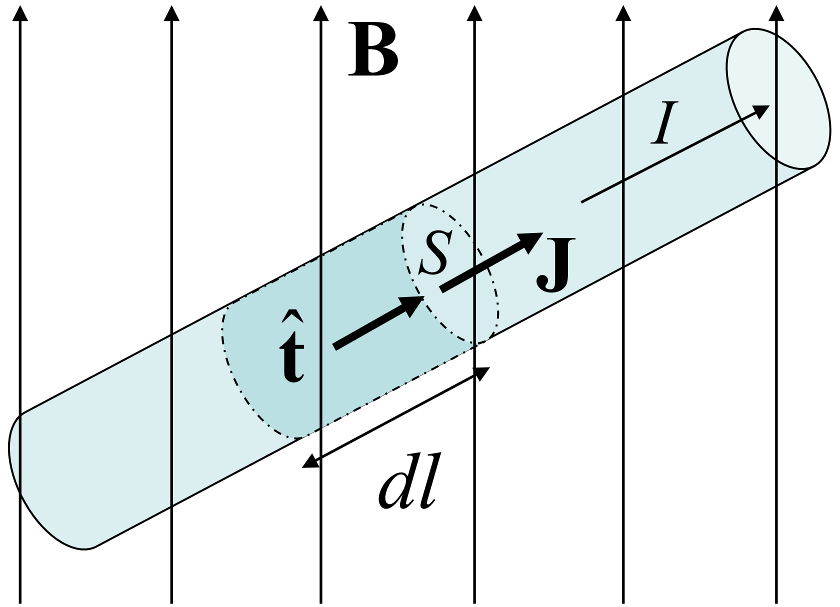
$$\mathbf{B} = \mu_0(\mathbf{H}_a + \mathbf{M}_s)$$

Magnetic force on a conductor with a current I in a field B

$$d\mathbf{l} = dl\hat{\mathbf{t}} \qquad \mathbf{S} = S\hat{\mathbf{t}}$$

$$\mathbf{J} = nq\mathbf{v} = nqv\hat{\mathbf{t}} \qquad \mathbf{J} = (I/S)\hat{\mathbf{t}}$$

n: density of "mobile" charged particles (m^{-3})
q: charge of the "moving" particle (C)



Force on the volume dlS:

$$\sum_i (q_i \mathbf{v}_i \times \mathbf{B}) = (\mathbf{v} \times \mathbf{B}) \sum_i q_i =$$
$$= (\mathbf{v} \times \mathbf{B}) nqdlS = nq\mathbf{v} \times \mathbf{B} dlS = \mathbf{J} \times \mathbf{B} dlS = I\hat{\mathbf{t}} \times \mathbf{B} dl$$

f: Force per unit of volume

F: Total force on the wire of lenght L

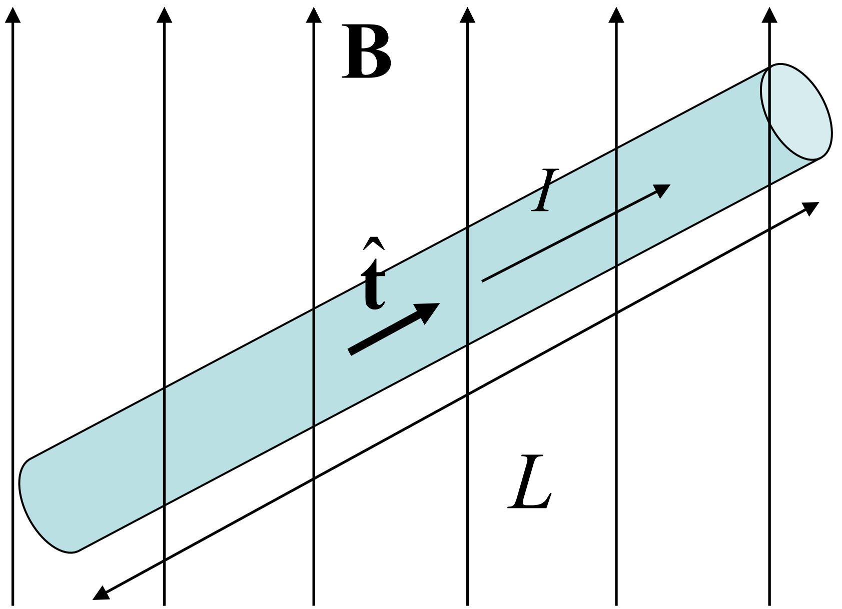
$$\mathbf{f} = nq\mathbf{v} \times \mathbf{B} = \mathbf{J} \times \mathbf{B} \qquad \Rightarrow \qquad \mathbf{F} = \int_V \mathbf{J} \times \mathbf{B}\, dV = I \int_L \hat{\mathbf{t}} \times \mathbf{B}\, dl = I \int_L d\mathbf{l} \times \mathbf{B}$$

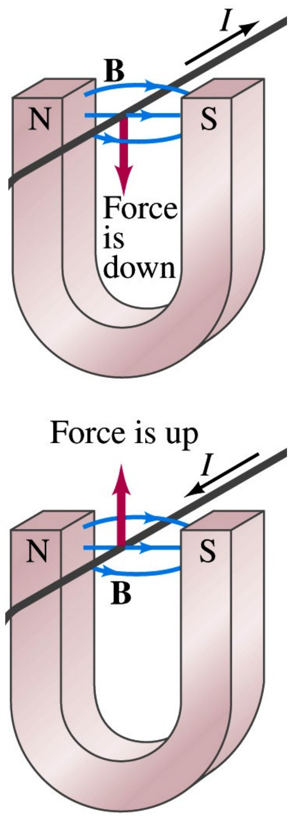
The force on a conductor is the Lorentz force on moving electric charges.

Example 1.: Straight wire of length L in a uniform field B

$$\mathbf{F} = I\int_0^L d\mathbf{l}\times\mathbf{B} = I\mathbf{L}\times\mathbf{B}$$

$$\mathbf{L} = L\hat{\mathbf{t}}$$





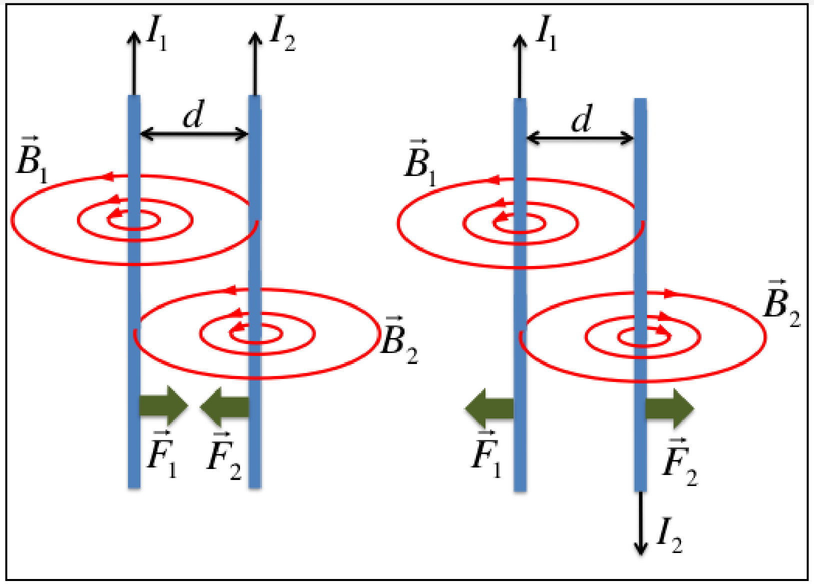
Example 2.: Closed circuit (of any shape) in a uniform B-field

$$\mathbf{F} = I\oint \hat{\mathbf{t}}\times\mathbf{B} = -I\mathbf{B}\times\oint d\mathbf{l} = 0$$

The resulting force on a closed circuit
of any shape in a uniform **B** field is zero

Example 3: Force between two infinite parallel wires

$$\mathbf{B} = \frac{\mu_0}{2\pi}\frac{I}{r}\hat{\boldsymbol{\varphi}} \qquad\qquad \mathbf{F} = q\mathbf{v}\times\mathbf{B}$$



$$\Rightarrow$$

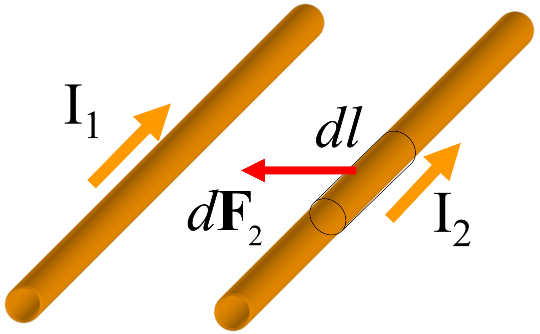
$$\mathbf{B}_1 = \frac{\mu_0}{2\pi}\frac{I_1}{d}\hat{\boldsymbol{\varphi}} \qquad\qquad \mathbf{B}_2 = -\frac{\mu_0}{2\pi}\frac{I_2}{d}\hat{\boldsymbol{\varphi}}$$

$$\Rightarrow$$

$$d\mathbf{F}_1 = \left(\sum_{dl} q_i\right)\mathbf{v}_1\times\mathbf{B}_2 = (nqSdl)\mathbf{v}_1\times\mathbf{B}_2 = I_1 d\mathbf{l}\times\mathbf{B}_2 = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{d}dl\hat{\mathbf{d}}$$

$$d\mathbf{F}_2 = \left(\sum_{dl} q_i\right)\mathbf{v}_2\times\mathbf{B}_1 = (nqSdl)\mathbf{v}_2\times\mathbf{B}_1 = I_2 d\mathbf{l}\times\mathbf{B}_1 = -\frac{\mu_0}{2\pi}\frac{I_1 I_2}{d}dl\hat{\mathbf{d}}$$

Forces on the element *dl*



Total forces on wires of length L

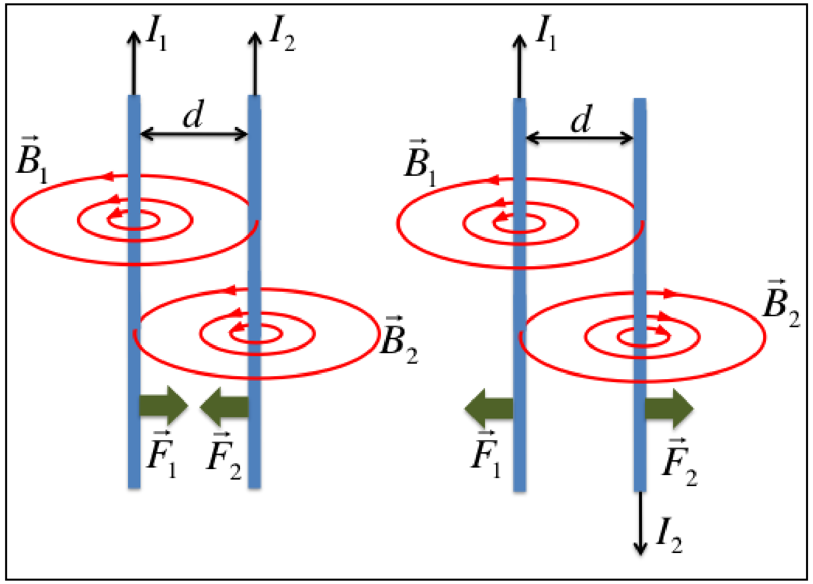
$$\mathbf{F}_1 \cong \int_0^L d\mathbf{F}_1 = \int_0^L \frac{\mu_0}{2\pi}\frac{I_1 I_2}{d}\,dl\,\hat{\mathbf{d}} = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{d}\,L\hat{\mathbf{d}}$$

$$\mathbf{F}_2 \cong \int_0^L d\mathbf{F}_2 = \int_0^L -\frac{\mu_0}{2\pi}\frac{I_1 I_2}{d}\,dl\,\hat{\mathbf{d}} = -\frac{\mu_0}{2\pi}\frac{I_1 I_2}{d}\,L\hat{\mathbf{d}}$$

Exemple:

$I_1 = I_2 = 100\,\mathrm{A}$; $d = 1$ cm ; $L = 1$ m

$F = 0.2\,\mathrm{N}$



Note: This experience was used to define the Ampere [A]

Galvanomagnetic effects in matter: simplified

In absence of a net molecular current in dV:

$$\mathbf{J}(\mathbf{x}) = \frac{1}{dV}\left(\sum_{i(free-charges)} q_i \mathbf{v}_i\right)$$

dV contains a large number of molecules around $\mathbf{x}$

If the free charges in dV have charge q and are moving with an mean velocity $\mathbf{v}$:

$$\boxed{\mathbf{J}(\mathbf{x}) = n(\mathbf{x})q\mathbf{v}(\mathbf{x})}$$

n: density of free charges (m^{-3})

In presence of **E** and **B** and of **scattering forces** we have:

$$\boxed{q\mathbf{E} + q\mathbf{v}\times\mathbf{B} = m\frac{\mathbf{v}}{\tau}}$$

Lorentz force and scattering forces

τ = relaxation time
= time to reach mean velocity
= half-time between collisions

$$\mathbf{J} = nq\mathbf{v}$$

$$q\mathbf{E} + q\mathbf{v}\times\mathbf{B} = m\frac{\mathbf{v}}{\tau}$$

Ohm law

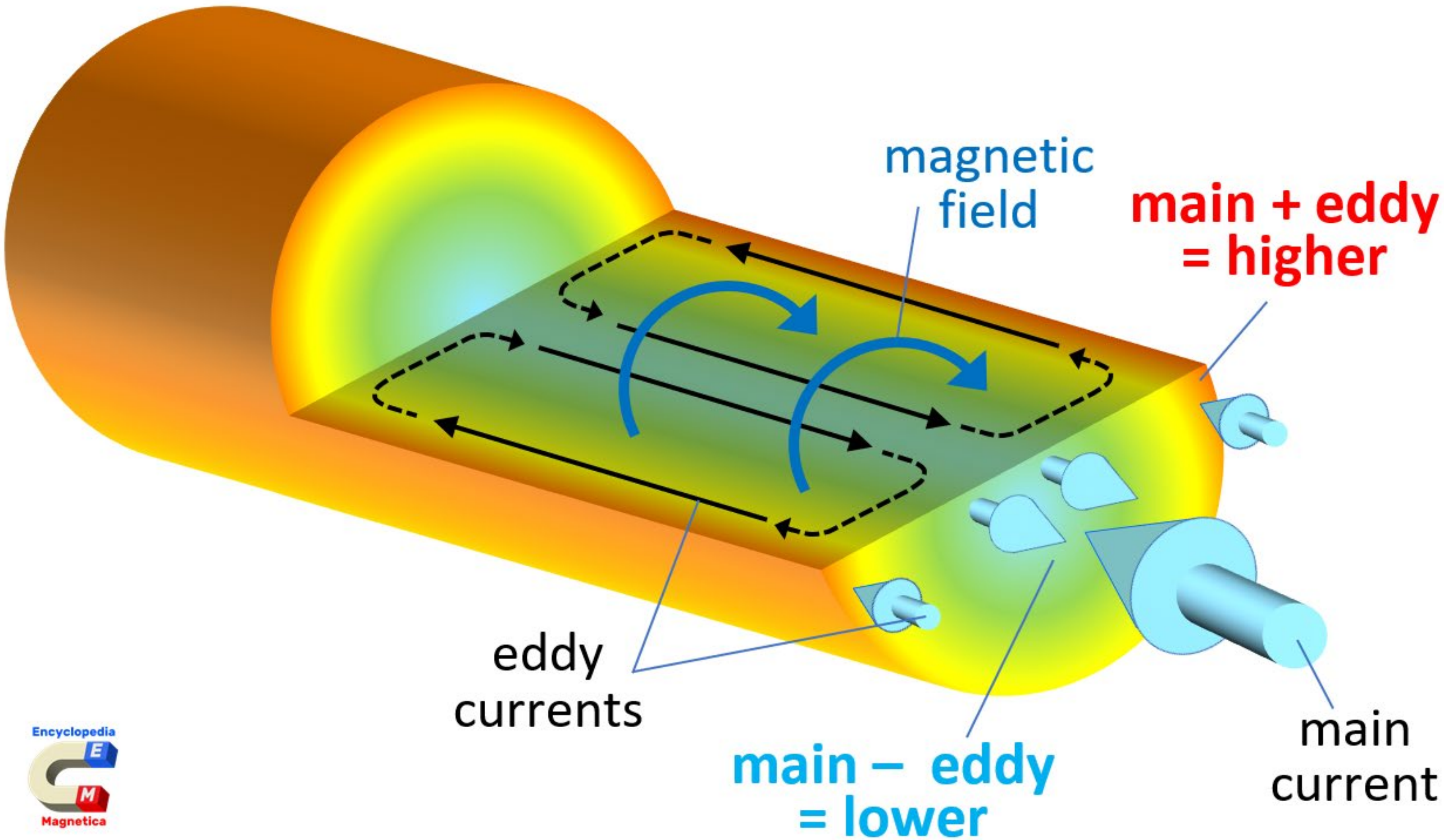
$$\mathbf{J} = \sigma\mathbf{E}$$

$$\sigma_0 \equiv \frac{nq^2\tau}{m}$$

$$\sigma \equiv \begin{pmatrix} 1/\sigma_0 & -\frac{1}{nq}B_z & \frac{1}{nq}B_y \\ \frac{1}{nq}B_z & 1/\sigma_0 & -\frac{1}{nq}B_x \\ -\frac{1}{nq}B_y & \frac{1}{nq}B_x & 1/\sigma_0 \end{pmatrix}^{-1}$$

Skin effect

Less current in center, more current at borders,
→ Skin effect



"Electromagnetic quasi-static" considerations

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{J} = \sigma \mathbf{E} = \sigma(-\nabla V - i\omega \mathbf{A})$$

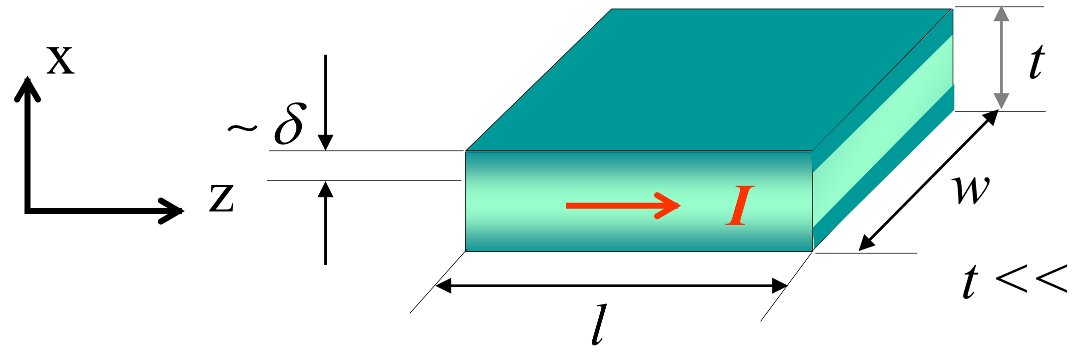
Harmonic assumption

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d\mathbf{x}'$$

$$\boxed{\mathbf{J}(\mathbf{x}) = -\sigma(\mathbf{x}) \cdot \nabla V - \overbrace{\frac{i\omega\sigma(\mathbf{x})\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \cdot d\mathbf{x}'}^{\text{Eddy currents}}}$$

Electric potential

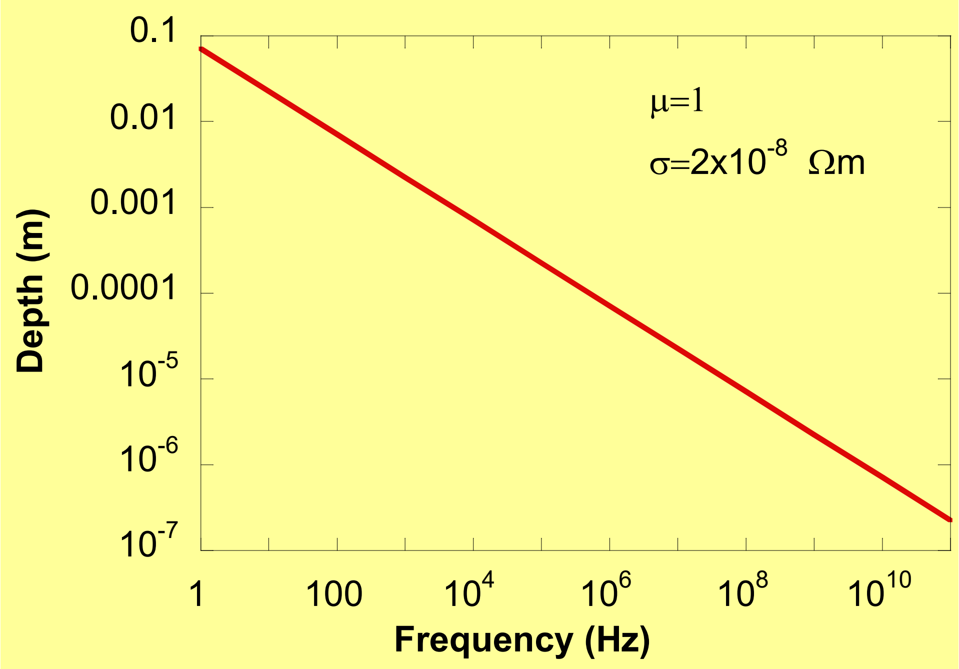
Skin effect: thin conducting plate



$$E_0 = \frac{V}{l}$$

$$t << l, w$$

$$\boxed{\delta \equiv \frac{\sqrt{2}}{\sqrt{\omega \mu_0 \mu \sigma}}}$$

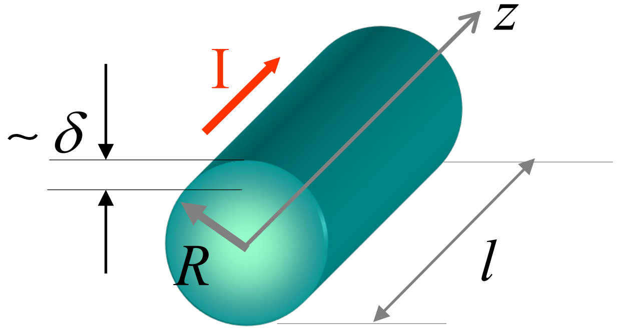


$$J_z(x) \cong \sigma E_0 \coth\left(\frac{\sqrt{2j}}{\delta} x\right)$$

$$Z = \frac{l}{2\sigma w \delta}(1+j)\frac{1}{\tanh\left((1+j)\frac{t}{2\delta}\right)}$$

For $\delta << t$: $R = \Re e(Z) \cong \dfrac{l}{2\sigma w \delta}$

Skin effect: cylindrical wire



$$E_0 = \frac{V}{l}$$

$$\boxed{\delta \equiv \frac{\sqrt{2}}{\sqrt{\omega \mu_0 \mu \sigma}}}$$

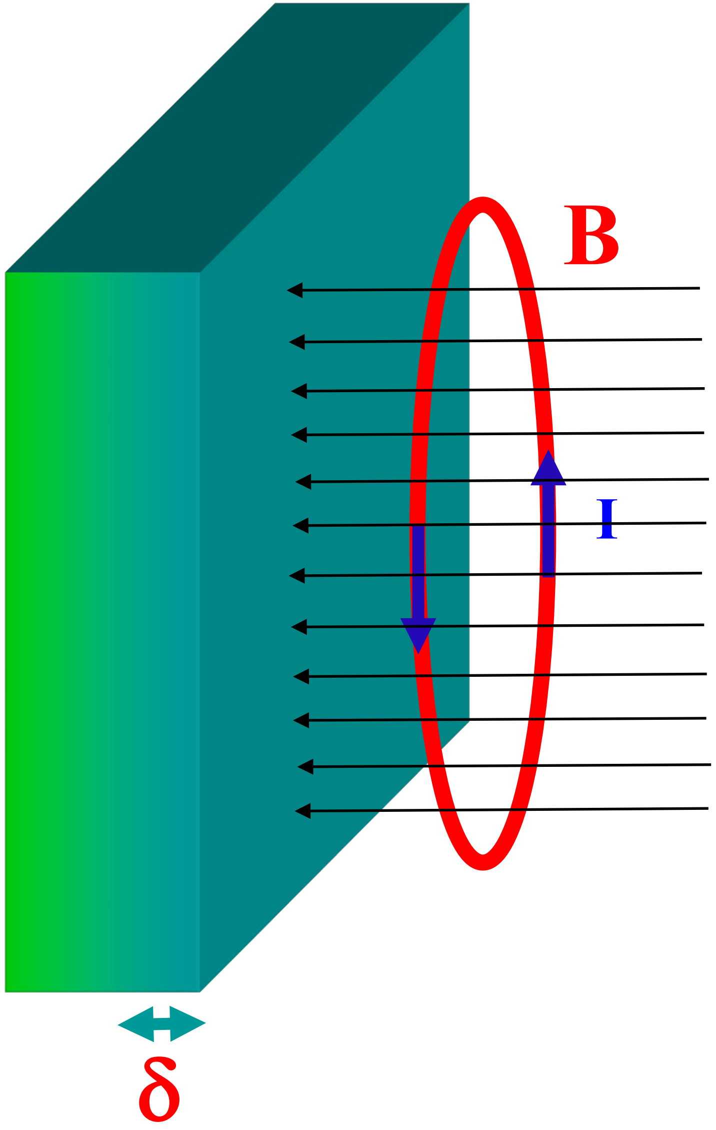
$$J_z(r) \cong \sigma E_0 J_0\left(\frac{\sqrt{2j}}{\delta} r\right) / J_0\left(\frac{\sqrt{2j}}{\delta} R\right)$$

$$Z = \frac{l}{\sigma 2\pi R\delta} \sqrt{2j} J_0\left(\frac{\sqrt{2j}}{\delta} R\right) / J_1\left(\frac{\sqrt{2j}}{\delta} R\right)$$

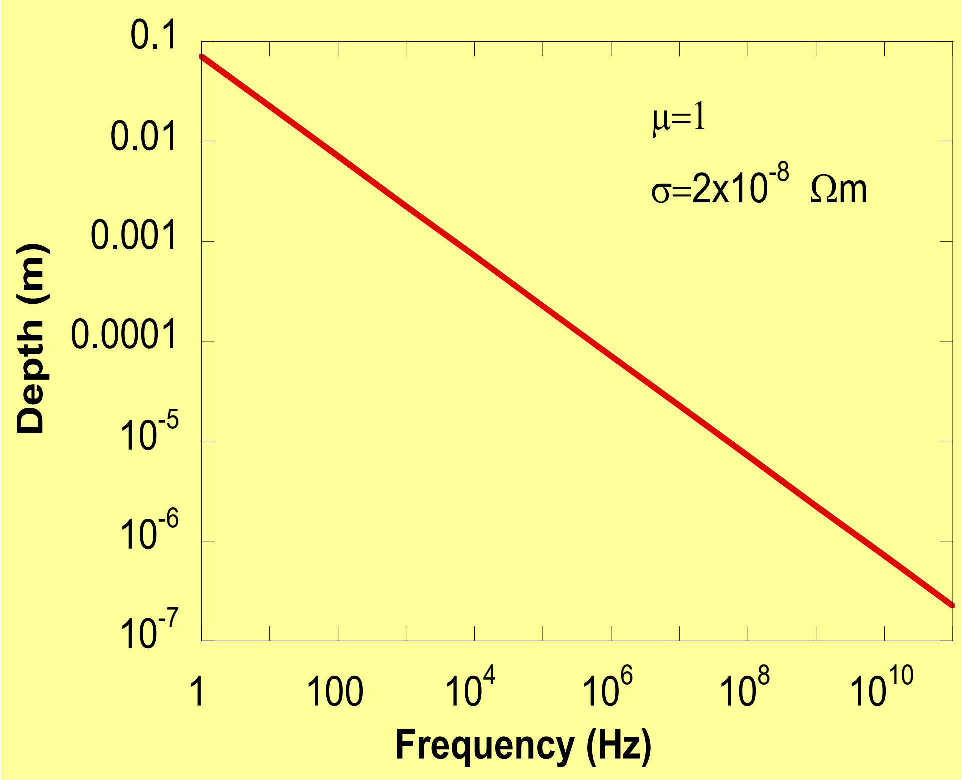
J_0 , J_1: Bessel functions

For $\delta << R$: $R = \Re e(Z) \cong \dfrac{l}{\sigma 2\pi R\delta}$

Skin effect and shielding

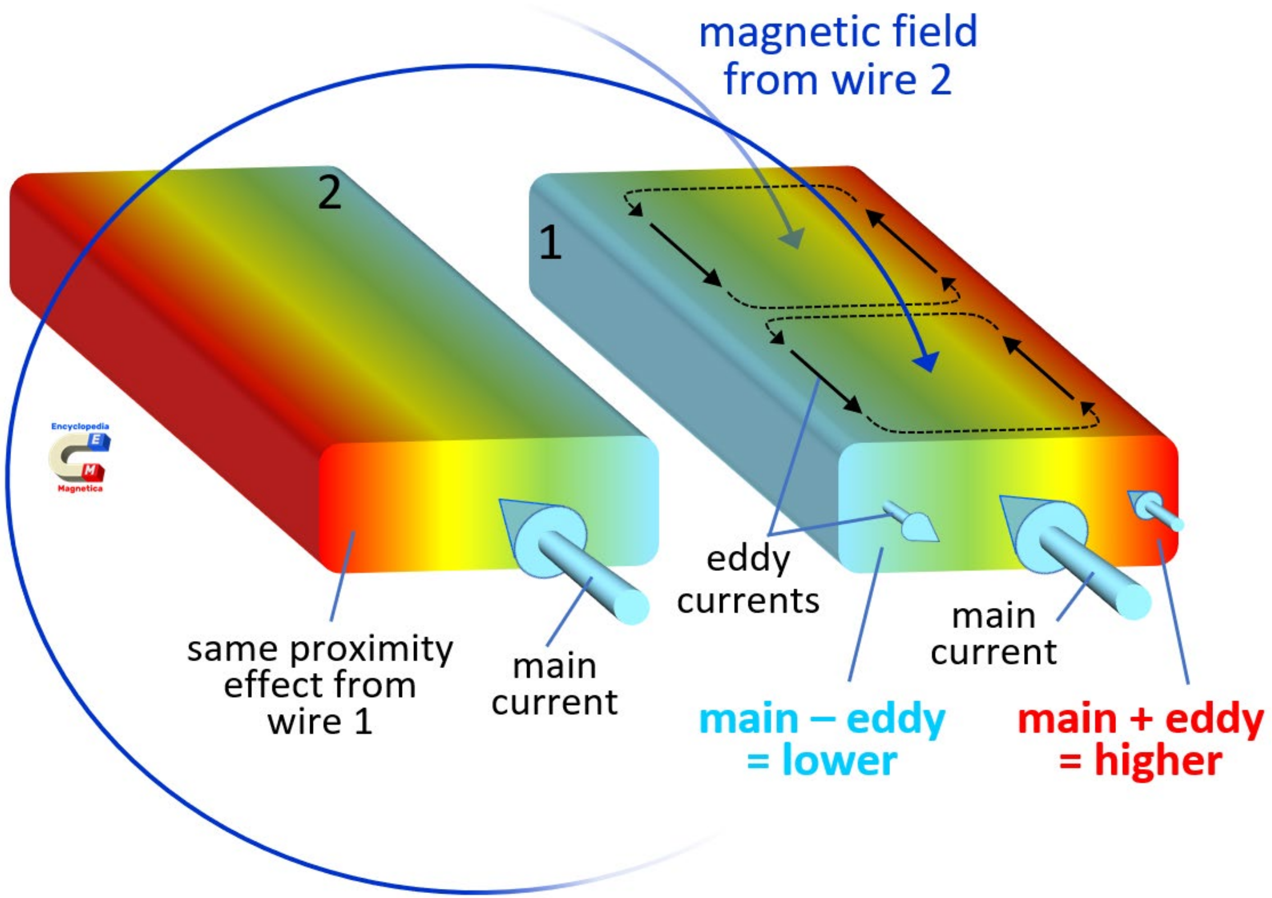


$$\delta \cong \frac{\sqrt{2}}{\sqrt{\omega \mu_0 \mu \sigma}}$$

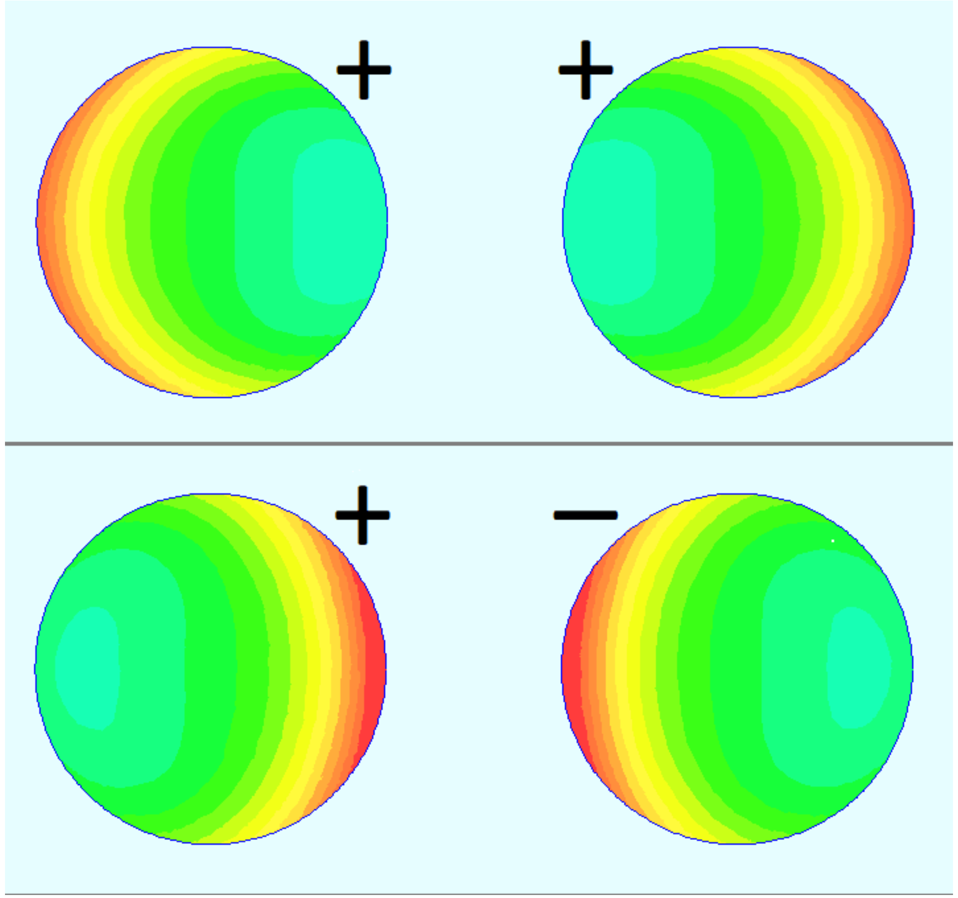


Proximity effect

Principle: same "physics" as skin effect



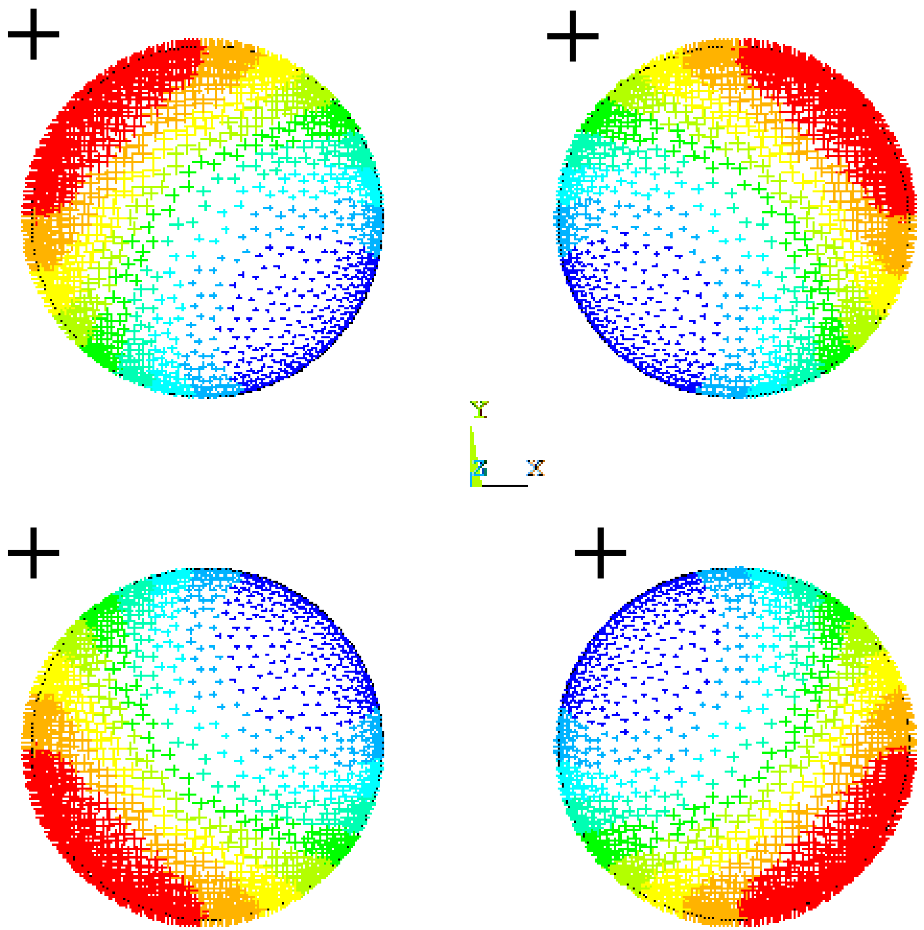
Proximity effect: two wires



Currents
in the **same** direction

Currents
in **opposite** direction

Proximity effect: four wires



Currents
in the **same** direction